MIXING ON RANK-ONE TRANSFORMATIONS

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ABSTRACT. We prove mixing on rank-one transformations is equivalent to "the uniform convergence of ergodic averages (as in the mean ergodic theorem) over subsequences of partial sums". In particular, all polynomial staircase transformations are mixing.

1. Introduction

1.1. Rank-One Transformations. Rank-one transformations are transformations "well-approximated" by a sequence of discrete spectrum transformations, so it was very surprising when in 1970 Ornstein [Orn72] showed the existence of rank-one *mixing* transformations. Rank-one mixing transformations are mixing of all orders [Kal84], [Ryz93] and enjoy other remarkable properties, see e.g. [Kin88]. Ornstein's construction was stochastic in nature: there is a class of rank-one transformations so that almost surely a transformation in that class is mixing; however, it did not yield a deterministic procedure for constructing one.

1.2. Staircase Transformations. A few years later, Smorodinsky conjectured that a specific rank-one transformation, the classical staircase transformation, is mixing. In 1992, Adams and Friedman [AF92] gave a deterministic algorithm involving a sequence of cutting and stacking constructions that produced a mixing rank-one transformation, and later Adams [Ada98] proved that Smorodinsky's conjecture is true. Informally, a staircase transformation is a cutting and stacking transformation with sequence $\{r_n\}$ of natural numbers such that at the n^{th} stage the n^{th} column or stack is cut into r_n subcolumns and "spacers" (see Section 3) are placed in a staircase fashion on the subcolumns before stacking, i.e., the number of spacers in each subsequent subcolumn is increased by 1. Adams showed that the resulting staircase transformation is mixing provided that $\frac{r_n^2}{h_n} \to 0$ as $n \to \infty$ (which also implies that the transformation is finite measure-preserving), where h_n denotes the number of levels, or height, of the n^{th} column. He then asked whether the mixing property holds for every finite measure-preserving staircase transformation simply under the assumption that $r_n \to \infty$. In 2003, Ryzhikov wrote the authors a short email stating that in 2000 he gave a lecture where he proved that all staircases are mixing (Theorem 1) [Ryz03] (giving a positive answer to Adams' question); the result presented here was developed independently of his work and indeed we did not receive his proof sketch until after sending a preprint of this paper to him in 2005. We would also like to thank Ryzhikov for asking a question that clarified our writing of the definition of polynomial staircase transformations. The application of our main theorem shows that polynomial staircase transformations

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are mixing (Theorem 4). Specializing to the case of linear polynomials shows that all staircase transformations are mixing.

1.3. Restricted Growth. The $\frac{r_n^2}{h_n} \to 0$ condition, a restriction on the asymptotic growth of the spacers relative to the column height, was generalized to all rank-one transformations and called "restricted growth" in [CS04]. The staircase transformation of Smorodinsky's conjecture is obtained when $r_n = n+1$; verifying that it satisfies the restricted growth condition is straightforward. In [CS04], the authors proved an equivalence between mixing and a condition on the spacer sequence for rank-one transformations with restricted growth. It followed that restricted growth rank-one transformations with the sequence of spacers given by a polynomial satisfying some general conditions (including the staircases of [Ada98]) are mixing. Ornstein's result also follows from that theorem.

1.4. Our Result. In this paper we lift the restricted growth condition from the theorems in [CS04]. We present a self-contained proof of a condition equivalent to mixing for rank-one transformations involving the uniform convergence of certain averages of partial sums of the spacer sequence. Staircase and polynomial staircase transformations satisfy this condition.

2. Mixing Properties

2.1. Dynamical Systems. For our study, **dynamical system** shall mean a standard probability measure space $(X, \mathcal{B}, \mu)$ and **transformation** $T : X \to X$ that is invertible, measurable and measure-preserving. Throughout the paper, (X, μ) is $[0, 1)$ under Lebesgue measure and $\mathcal{B}$ is the algebra of Lebesgue measurable subsets.

2.2. Mixing. A transformation T is **mixing** when for all $A, B \in \mathcal{B}$,

$$\lim_{n\to\infty} \mu(T^n(A) \cap B) - \mu(A)\mu(B) = 0;$$

$\{t_n\}$ is a **mixing sequence** (with respect to T) when for all $A, B \in \mathcal{B}$,

$$\lim_{n\to\infty} \mu(T^{t_n}(A) \cap B) - \mu(A)\mu(B) = 0.$$

2.3. Ergodicity. A transformation T is **ergodic** when for all $A \in \mathcal{B}$, if $T^{-1}(A) = A$ then $\mu(A) = 0$ or $\mu(A) = 1$. The **mean (von Neumann) ergodic theorem** states that T is ergodic if and only if for all $B \in \mathcal{B}$,

$$\lim_{n\to\infty} \int \Big|\frac{1}{n}\sum_{j=0}^{n-1} \chi_B \circ T^{-j} - \mu(B)\Big| d\mu = 0.$$

(χ_B being the characteristic function of the set B.). A transformation T is **totally ergodic** when for any $\ell \in \mathbb{N}, \ell \neq 0$, the transformation T^ℓ is ergodic. (We use the notation $\mathbb{N} = \{0, 1, 2, \ldots\}$ for the natural numbers and $\mathbb{Z}_N = \{0, 1, \ldots, N-1\}$ for the N-element subset of $\mathbb{N}$ with the usual order).

2.4. Ergodic Sequences. Unless otherwise stated, the term **sequence** shall mean sequence in $\mathbb{N}$ that is strictly increasing. A sequence $\{a_n\}$ is an **ergodic sequence** (with respect to a transformation T) when for all $B \in \mathcal{B}$,

$$\lim_{n\to\infty} \int \Big|\frac{1}{n}\sum_{j=0}^{n-1} \chi_B \circ T^{-a_j} - \mu(B)\Big| d\mu = 0.$$

2.5. Power Ergodicity. We introduce the concept of power ergodicity, all powers of an ergodic transformation being "uniformly" ergodic in the sense that the ergodic averages converge uniformly (to the projection onto the constants). Earlier results on specific rank-one mixing used precursors to this notion, including the uniform Cesàro property used in [AF92] (and implicitly in [Ada98]) and power uniform ergodicity in [CS04].

Definition 2.1. *A transformation T is* **power ergodic** *when for all $B \in \mathcal{B}$,*

$$\lim_{n\to\infty} \sup_{k\in\mathbb{N}} \int \Big|\frac{1}{n}\sum_{j=0}^{n-1} \chi_B \circ T^{-jk} - \mu(B)\Big| d\mu = 0.$$

3. Rank-One Transformations

3.1. Cutting and Stacking. Begin with $[0,1)$, the only "level" in the initial "column". "Cut" it into r_0 "sublevels", pieces of equal length: $[0,\frac{1}{r_0})$, $[\frac{1}{r_0},\frac{2}{r_0})$, $\ldots$, $[\frac{r_0-1}{r_0},1)$. Place an interval of the same length "above" $[\frac{1}{r_0},\frac{2}{r_0})$, i.e., place $[1,\frac{r_0+1}{r_0})$ above $[\frac{1}{r_0},\frac{2}{r_0})$. Likewise, place j "spacer" sublevels above each piece. Now, "stack" the resulting subcolumns from left to right by placing $[0,\frac{1}{r_0})$ at the bottom, $[\frac{1}{r_0},\frac{2}{r_0})$ above it, the spacer level above that, $[\frac{2}{r_0},\frac{3}{r_0})$ above the spacer and so on, ending with the topmost of the $r_0 - 1$ spacers. This stack of $h_1 = r_0 + \sum_{j=0}^{r_0-1} j$ levels (of length $\frac{1}{r_0}$), the second column, defines a map $T_0 : [0, 1 + \frac{1}{r_0}\sum_{j=0}^{r_0-1} j - \frac{1}{r_0}) \to [\frac{1}{r_0}, 1 + \frac{1}{r_0}\sum_{j=0}^{r_0-1} j)$ that sends points directly up one level.

Repeat the process: cut the entire new column into r_1 subcolumns of equal width $\frac{1}{r_0 r_1}$, preserving the stack map on each subcolumn; place j spacers (intervals not yet in the space the same width as the subcolumns) above each subcolumn ($j \in \mathbb{Z}_{r_1}$); and stack the resulting subcolumns from left to right. Our new column defines a map T_1 that agrees with T_0 where it is defined and extends it to all but the topmost spacer of the rightmost subcolumn. Iterating this process leads to a transformation T defined on all but a Lebesgue measure zero set.

The transformations obtained in this manner are called **staircase transformations**. More generally, one may place $s_{n,j}$ spacers above the j^{th} subcolumn at the n^{th} stage in place of the j spacers above. A transformation created by *cutting and stacking* as just described (with a single column resulting from each iteration) is a **rank-one transformation**. The reader is referred to [Fer97] and [Fri70] for more details. Rank-one transformations are measurable and measure-preserving under Lebesgue measure, and are completely defined by the doubly-indexed sequence $\{s_{n,j}\}_{\{r_n\}}$ where at the n^{th} step we cut into r_n pieces and place $s_{n,j}$ spacers above each subcolumn (for staircase transformations, $s_{n,j} = j$). This $\{s_{n,j}\}_{\{r_n\}}$ is the **spacer sequence** for the transformation and $\{r_n\}$ is the **cut sequence**. The **height sequence** $\{h_n\}$ is the number of levels in each column: $h_0 = 1$ and $h_{n+1} = r_n h_n + \sum_{j=0}^{r_n-1} s_{n,j}$. It is well-known (and left to the reader) that if $\liminf r_n < \infty$ then the transformation will be partially rigid hence cannot be mixing. We shall assume from here on that $\lim r_n = \infty$.

We write $I_{n,i}$ to denote the i^{th} level in the n^{th} stack ($i \in \mathbb{Z}_{h_n}$) where $I_{n,0}$ is the bottom level and $T(I_{n,i}) = I_{n,i+1}$ and write $C_n = \bigcup_{i=0}^{h_n-1} I_{n,i}$ to denote the n^{th} column and $S_n = C_{n+1} \setminus C_n$ to denote the spacers added. We write $I_{n,i}^{[j]}$ for the j^{th} sublevel of the i^{th} level of the n^{th} column, i.e., $I_{n,0}^{[0]}$ is the leftmost sublevel of the bottom level ($I_{n,0}^{[0]} = I_{n+1,0}$ becomes the bottom level of the next column). Note that T is defined on a finite measure space if and only if $\sum_{n=0}^{\infty} \mu(S_n) < \infty$ and in that case T is isomorphic to the transformation defined on $[0,1)$ obtained by cutting and stacking in the same fashion as T but beginning with $C_0 = [0, \frac{1}{K})$ where K is the measure of the space the original T is defined on.

4. Mixing on Staircase Transformations

We shall first prove directly that staircase transformations are mixing to illustrate the techniques used. Staircases are the simplest of the mixing rank-one transformations and serve as a model for the general case. As mentioned in the introduction, the following theorem was proved by Adams under a growth restriction [Ada98] and announced by Ryzhikov in 2000 (unpublished).

Theorem 1. *Let T be a staircase transformation. Then T is mixing.*

In what follows we shall first outline the ideas and then present the formal proof. Consider a level I in the n^{th} column defining a rank-one transformation that is at least r_n above the bottom. Look at T^{h_n} applied to the j^{th} sublevel of I. There are j spacers added above that subcolumn so T^{h_n} will map the j^{th} sublevel of I to the $j+1^{th}$ sublevel of the level j below I. This means that T^{h_n} maps I to a progression of $\frac{1}{r_n}$ sized parts of r_n consecutive levels. So the characteristic function of $T^{h_n}(I)$ is in fact an average of characteristic functions of consecutive levels which is to say that it is of the form $\frac{1}{r_n}\sum_{j=0}^{r_n-1} \chi_I \circ T^{-j}$. The ergodicity of T then guarantees that this quantity tends to zero so in fact the sequence $\{h_n\}$ will be mixing.

In what follows, this idea of turning a sequence into an ergodic type average will be the primary ingredient. We first establish that for an arbitrary sequence, the mixing behavior is controlled by specific ergodic type averages. We then prove that for staircase transformations, any ergodic type average along arithmetic progressions tends to zero.

The following Lemma formalizes our discussion of sublevels above:

Lemma 4.1. *Let $p \in \mathbb{N}, i \in \mathbb{Z}_{h_p}, j \in \mathbb{Z}_{r_p}$, and B a union of levels in C_p. Then*

$$(i) \quad I_{p,i} = \bigcup_{j=0}^{r_p-1} I_{p,i}^{[j]};$$

$$(ii) \quad T^{kh_p+jk+\frac{1}{2}k(k-1)} I_{p,i}^{[j]} = I_{p,i}^{[j+k]}; \text{ and}$$

$$(iii) \quad \mu(I_{p,i}^{[j]} \cap B) = \frac{1}{r_p}\mu(I_{p,i} \cap B).$$

Proof. (i) is immediate from the construction of rank-one transformations; (ii) follows from k applications of the fact that $T^{h_p+j} I_{p,i}^{[j]} = I_{p,i}^{[j+1]}$; and for (iii), B is a union of levels in C_p so $I_{p,i} \subseteq B$ or $I_{p,i} \cap B = \emptyset$. □

The first step is our proof of mixing is the height sequence:

Proposition 4.2. *Let T be a staircase transformation with height sequence $\{h_n\}$. Then $\{h_n\}$ is mixing with respect to T.*

Proof. This follows from the Lemma after first dropping the bottom r_n levels and the rightmost column (which have vanishing measure) and then using that $T^{h_n}(I)$

will be an ergodic type average of the powers of T. To make this formal, it suffices to show that $\mu(T^{h_n}A\cap B)\to\mu(A)\mu(B)$ for A,B unions of levels in C_N for arbitrary $N\in\mathbb{N}$ since levels generate measurable sets. For any $n\geq N$ and $r_n\leq i<h_n$ ($\{r_n\}$ being the cut sequence for T), using Lemma 4.1 parts (i),(ii) and (iii) in that order,

$$\begin{aligned}
&\big|\mu(T^{h_n}I_{n,i}\cap B)-\mu(I_{n,i})\mu(B)\big|\\
&\quad=\big|\sum_{j=0}^{r_n-1}\mu(T^{h_n}I_{n,i}^{[j]}\cap B)-\mu(I_{n,i}^{[j]})\mu(B)\big|\\
&\quad\leq\big|\sum_{j=0}^{r_n-2}\mu(T^{-j}I_{n,i}^{[j+1]}\cap B)-\mu(I_{n,i}^{[j+1]})\mu(B)\big|+\mu(I_{n,i}^{[r_n-1]})\\
&\quad=\big|\sum_{j=0}^{r_n-2}\mu(I_{n,i-j}^{[j+1]}\cap B)-\mu(I_{n,i-j}^{[j+1]}\mu(B)\big|+\frac{1}{r_n}\mu(I_{n,i})\\
&\quad=\big|\frac{1}{r_n}\sum_{j=0}^{r_n-2}\mu(I_{n,i-j}\cap B)-\mu(I_{n,i-j})\mu(B)\big|+\frac{1}{r_n}\mu(I_{n,i})\\
&\quad\leq\big|\frac{1}{r_n}\sum_{j=0}^{r_n-1}\mu(T^{-j}I_{n,i}\cap B)-\mu(I_{n,i})\mu(B)\big|+\frac{2}{r_n}\mu(I_{n,i}).
\end{aligned}$$

Now A is a union of levels in C_N hence in C_n so

$$\begin{aligned}
&\big|\mu(T^{h_n}A\cap B)-\mu(A)\mu(B)\big|\\
&\leq\sum_{i=0}^{h_n-1}\big|\mu(T^{h_n}I_{n,i}\cap B)-\mu(I_{n,i})\mu(B)\big|\\
&\leq\sum_{i=r_n}^{h_n-1}\big|\frac{1}{r_n}\sum_{j=0}^{r_n-1}\mu(T^{-j}I_{n,i}\cap B)-\mu(I_{n,i})\mu(B)\big|+\sum_{i=r_n}^{h_n-1}\frac{2}{r_n}\mu(I_{n,i})+\sum_{i=0}^{r_n-1}\mu(I_{n,i})\\
&\leq\int\big|\frac{1}{r_n}\sum_{j=0}^{r_n-1}\chi_B\circ T^{-j}-\mu(B)\big|d\mu+\frac{2}{r_n}+\frac{r_n}{h_n}.
\end{aligned}$$

By assumption, $r_n\to\infty$, and since T is defined on a finite measure space, $\frac{r_n}{h_n}\to 0$ so the ergodicity of T implies the result. □

The following technical Lemma will be used in combination with Lemma 4.1 part (ii) to obtain that if we look at a block of ℓ consecutive sublevels of some level I then after applying T^{kh_n} and dropping the levels at the bottom of the column we are left with an ergodic type average of powers of T^k exactly as we were for h_n above but weighted by the size of the block $\frac{\ell}{r_n}$.

Lemma 4.3. *For $p,k\in\mathbb{N}$, $\ell\in\mathbb{Z}_{r_p}$ and $c\in\mathbb{Z}_{r_p-\ell}$,*

$$\sum_{i=0}^{h_p-1}\big|\sum_{j=0}^{\ell-1}\mu(T^{-jk}I_{p,i}^{[j+c]}\cap B)-\mu(I_{p,i}^{[j]})\mu(B)\big|\leq\frac{\ell}{r_p}\int\big|\frac{1}{\ell}\sum_{j=0}^{\ell-1}\chi_B\circ T^{-jk}-\mu(B)\big|+\frac{\ell^2k}{r_ph_p}.$$

Proof. For $i\geq\ell k$, $\mu(T^{-jk}I_{p,i}^{[j+c]}\cap B)=\frac{1}{r_p}\mu(T^{-jk}I_{p,i}\cap B)$ by Lemma 4.1 part (iii). For $i<\ell k$,
$\big|\sum_{j=0}^{\ell-1}\mu(T^{-jk}I_{p,i}^{[j+c]}\cap B)-\mu(I_{p,i}^{[j+c]})\mu(B)\big|\leq\ell\mu(I_{p,i}^{[0]})\leq\frac{\ell}{r_ph_p}$ since $\mu(C_p)\leq 1$. □

Consider now the ergodic type average of r_n powers of $T^{k(n)}$, that is, let k vary with n. It is clear from the construction of rank-one transformations by columns that if $k(n)h_p \le t_n \le (k(n)+1)h_p$ then the behavior of T^{t_n} is controlled by that of $T^{k(n)h_p}$ and $T^{(k(n)+1)h_p}$. The following proposition states that if we know a priori that the ergodic type average of the powers of $T^{k(n)}$ tend to zero and furthermore that $k(n)$ is small enough that we may drop the bottom $k(n)$ levels (these levels have vanishing measure) then following the same procedure as above, $T^{k(n)h_p}$ will be mixing and so then will the sequence.

Definition 4.1. *In the context of a rank-one transformation with heights $\{h_n\}$ and cuts $\{r_n\}$, given $t \in \mathbb{N}$, the **unique** p **and** k **for** t are the unique numbers $p, k \in \mathbb{N}$ such that $h_p \le t < h_{p+1}$ and $kh_p \le t < (k+1)h_p$.*

Note that since $k < \frac{h_{p+1}}{h_p}$ and by the finite measure-preserving property $\frac{h_{p+1}}{h_p r_p} = \frac{\mu(C_{p+1})}{\mu(C_p)} \to 1$, we have that $\frac{k}{r_p} < \frac{h_{p+1}}{h_p r_p} \to 1$.

In the sequel, when we say "choose p and k" we shall mean the above construction. In particular, we shall often assume that $k \le r_p$ and leave to the reader to verify that in fact knowing $k \le (1+\epsilon)r_p$ for small ϵ suffices.

Proposition 4.4. *Let T be a staircase transformation with cut sequence $\{r_n\}$ and height sequence $\{h_n\}$ and let $\{t_n\}$ be a sequence. Choose $p(n)$ and $k(n)$ such that $h_{p(n)} \le t_n < h_{p(n)+1}$ and $k(n)h_{p(n)} \le t_n < (k(n)+1)h_{p(n)}$. If $\frac{r_{p(n)}k(n)}{h_{p(n)}} \to 0$ and for any $B \in \mathcal{B}$,*

$$\lim_{n\to\infty} \int \Big| \frac{1}{r_{p(n)}} \sum_{j=0}^{r_{p(n)}-k(n)-1} \chi_B \circ T^{-jk(n)} - \mu(B) \Big| d\mu = 0$$

and likewise replacing $k(n)$ by $k(n)+1$, then $\{t_n\}$ is mixing with respect to T.

Proof. The idea here is to write $k(n)$ and h_p as above, the proposition then follows from the fact that $T^{k_n h_p}$ sends a level in C_p to $\frac{1}{r_p}$ parts of a progression of $r_p - k_n$ levels that are all k_n apart so the ergodic type average of the powers of T^{k_n} is exactly the condition needed.

We now make this precise. Here, and in the proofs in the sequel, we shall sometimes drop the explicit dependence on n: we write $p = p(n)$ and $k = k(n)$. Pick $m = m(n)$ such that $t_n = kh_p + m$ where $0 \le m < h_p$. Let A, B be unions of levels in C_N for some $N \in \mathbb{N}$. For n such that $p \ge N$, as in Proposition 4.2,

$$\begin{aligned}
\big|\mu(T^{t_n}A \cap B) - \mu(A)\mu(B)\big| &\le \sum_{i=0}^{h_p-1} \big|\mu(T^{kh_p+m} I_{p,i} \cap B) - \mu(I_{p,i})\mu(B)\big| \\
&= \sum_{i=0}^{h_p-m-1} \big|\mu(T^{kh_p} I_{p,i+m} \cap B) - \mu(I_{p,i+m})\mu(B)\big| \\
&\quad + \sum_{i=h_p-m}^{h_p-1} \big|\mu(T^{(k+1)h_p} I_{p,i+m-h_p} \cap B) - \mu(I_{p,i+m-h_p})\mu(B)\big| \\
&\le \sum_{i=0}^{h_p-1} \big|\mu(T^{kh_p} I_{p,i} \cap B) - \mu(I_{p,i})\mu(B)\big|
\end{aligned}$$

$$+\sum_{i=0}^{h_p-1}\left|\mu(T^{(k+1)h_p}I_{p,i}\cap B)-\mu(I_{p,i})\mu(B)\right|.$$

Now by Lemma 4.1 part (i) and the triangle inequality,

$$\sum_{i=0}^{h_p-1}\left|\mu(T^{kh_p}I_{p,i}\cap B)-\mu(I_{p,i})\mu(B)\right|$$
$$\leq\sum_{i=0}^{h_p-1}\Big|\sum_{j=0}^{r_p-k-1}\mu(T^{kh_p}I_{p,i}^{[j]}\cap B)-\mu(I_{p,i}^{[j]})\mu(B)\Big|$$
$$+\sum_{i=0}^{h_p-1}\sum_{j=r_p-k}^{r_p-1}\left|\mu(T^{kh_p}I_{p,i}^{[j]}\cap B)-\mu(I_{p,i}^{[j]})\mu(B)\right|$$

and $T^{kh_p}I_{p,i}^{[j]}=T^{(k+j)h_p+\frac{1}{2}j(j-1)}I_{p+1,i}$ so for $j\geq r_p-k$ we have $(k+j)h_p+\frac{1}{2}j(j-1)\geq r_ph_p=h_{p+1}-\frac{1}{2}r_p(r_p-1)$ so in fact $T^{kh_p}I_{p,i}^{[j]}=T^{h_{p+1}}I_{p,a}^{[b]}$ for some a and b and $I_{p,a}^{[b]}$ is a level in C_{p+1} hence

$$\sum_{i=0}^{h_p-1}\sum_{j=r_p-k}^{r_p-1}\left|\mu(T^{kh_p}I_{p,i}^{[j]}\cap B)-\mu(I_{p,i}^{[j]})\mu(B)\right|$$
$$\leq\sum_{i=0}^{h_{p+1}-1}\left|\mu(T^{h_{p+1}}I_{p+1,i}\cap B)-\mu(I_{p+1,i})\mu(B)\right|+\frac{1}{2}r_p(r_p-1)\mu(I_{p+1,0})$$

which approaches zero since $\frac{1}{2}r_p(r_p-1)\mu(I_{p+1,0})=\mu(S_p)\to 0$ and as in the proof of Proposition 4.2.

For $\frac{1}{2}k(k-1)\leq i<h_p$, using Lemma 4.1 part (ii), we have that $T^{kh_p}I_{p,i}^{[j]}=T^{-jk}I_{p,i-\frac{1}{2}k(k-1)}^{[j+k]}$ so by Lemma 4.3,

$$\sum_{i=0}^{h_p-1}\Big|\sum_{j=0}^{r_p-k-1}\mu(T^{kh_p}I_{p,i}^{[j]}\cap B)-\mu(I_{p,i}^{[j]})\mu(B)\Big|$$
$$\leq\int\Big|\frac{1}{r_p}\sum_{j=0}^{r_p-k-1}\chi_B\circ T^{-jk}-\mu(B)\Big|d\mu+\frac{(r_p-k)^2k}{r_ph_p}+\frac{1}{2}k(k-1)\frac{r_p-k}{r_p}\mu(I_{p,0}).$$

Now $\frac{(r_p-k)^2k}{r_ph_p}\leq\frac{r_pk}{h_p}\to 0$ and $\frac{1}{2}k(k-1)\frac{r_p-k}{r_p}\mu(I_{p,0})\leq\frac{k^2}{h_p}\leq\frac{kr_p}{h_p}\to 0$ as $n\to\infty$ by hypothesis and the integral goes to zero by the final hypothesis. The above repeats similarly for the $k+1$ part. □

We now examine the case of a sequence where the $k(n)$ derived as above is not small enough to simply drop the bottommost levels:

Proposition 4.5. *Let T be a staircase transformation with cut sequence $\{r_n\}$ and height sequence $\{h_n\}$ and let $\{t_n\}$ be a sequence. Choose $p(n)$ and $k(n)$, as usual, such that $k(n)h_{p(n)}\leq t_n<(k(n)+1)h_{p(n)}$. If there exists a sequence $\{\ell(n)\}$ such that $\ell(n)\to\infty$, $\frac{\ell(n)k(n)}{h_{p(n)}}\to 0$ and $\frac{\ell(n)}{r_{p(n)}}\to 0$ and such that for any $\alpha(n,q)\in\mathbb{N}$*

indexed over $0 \leq q < Q_n = \lceil \frac{r_{p(n)}-k(n)}{\ell(n)} \rceil$ *with* $\limsup_n \sup_q \frac{\alpha(n,q)}{k(n)} < 1$ *and* $\alpha(n,q) \leq \alpha(n,q+1) \leq \alpha(n,q)+1$,

$$\lim_{n\to\infty} \frac{1}{Q_n} \sum_{q=0}^{Q_n-1} \int \Big| \frac{1}{\ell(n)} \sum_{j=0}^{\ell(n)-1} \chi_B \circ T^{-j(k(n)+2-\alpha(n,q))} - \mu(B) \Big| d\mu = 0$$

then $\{t_n\}$ *is mixing with respect to* T.

Proof. The idea here is to break the sublevels up into blocks of size small enough that we can drop the bottommost levels for each block separately and then apply our above methods to each block. Our Lemma on blocks of sublevels tells us each will be weighted by the size of the block so if each block's ergodic type average tends to zero then the entire quantity will as well.

The reason we have been dropping the bottommost levels above is since the behavior of the sublevels of those bottom levels is not to form an evenly spaced progression on the levels but in fact to form two or more such progressions by "coming back through the top" of the column. This coming back through the top can be avoided if we can ensure that the $k(n)$ we are using on a given block has the property that $jk(n)$ is small compared to h_n for $j = 0, \ldots, \ell$ where ℓ is the size of the block. The following proposition states and proves this formally: if the blocks we use are of size $\ell(n)$ and for any $k(n) - \alpha$ corresponding to the number of h_n's we need to remove when examining each block (α depends on the block), then if the ergodic type average tends to zero, the original sequence will be mixing. The conditions required on the $\ell(n)$ are necessary to perform the step of dropping the bottommost levels for each block (independently).

Now we make this precise: as before we shall drop the explicit mention of n and write $p = p(n)$, $k = k(n)$, $\ell = \ell(n)$, $Q = Q_n$ and $\alpha(\cdot) = \alpha(n, \cdot)$. Let B be a union of levels in C_N for some $N \in \mathbb{N}$. Then, as in the proof of Proposition 4.4, it suffices to show that

$$\sum_{i=0}^{h_p-1} \Big| \sum_{j=0}^{r_p-k-1} \mu(T^{kh_p} I_{p,i}^{[j]} \cap B) - \mu(I_{p,i}^{[j]})\mu(B) \Big| \to 0$$

and similarly for $k+1$. Now

$$\sum_{i=0}^{h_p-1} \Big| \sum_{j=0}^{r_p-k-1} \mu(T^{kh_p} I_{p,i}^{[j]} \cap B) - \mu(I_{p,i}^{[j]})\mu(B) \Big|$$
$$\leq \sum_{i=0}^{h_p-1} \sum_{q=0}^{Q-1} \Big| \sum_{j=0}^{\ell-1} \mu(T^{kh_p} I_{p,i}^{[j+q\ell]} \cap B) - \mu(I_{p,i}^{[j+q\ell]})\mu(B) \Big| + \sum_{i=0}^{h_p-1} \ell\mu(I_{p,i}^{[0]})$$

and $\sum_{i=0}^{h_p-1} \ell\mu(I_{p,i}^{[0]}) = \frac{\ell}{r_p}\mu(C_p) \to 0$ by hypothesis. For $0 \leq q < Q$, choose $\alpha(q) = \alpha(n,q)$ minimally such that $q\ell k + \frac{1}{2}k(k-1) < \alpha(q)(h_p + q\ell + k - \frac{1}{2}) - \frac{1}{2}\alpha(q)^2$. This reasoning for this choice of α is that T^{kh_p} acts like an average of $T^{-j(k-\alpha)}$ due to the spacer levels being added being not small compared to h_p, that is, T^{kh_p} would push levels through k times except that the spacers remove α of those times; we have chosen α so that $T^{kh_p} I_{p,i}^{[q\ell]} = T^w I_{p,i}^{[q\ell+k-\alpha(q)]}$ such that $w > 0$ and α is minimal. The $\alpha(q)$ are clearly nondecreasing and since $\frac{\ell r_p}{h_p} \to 0$ (by hypothesis) we have $\alpha(q+1) - \alpha(q) \leq 1$ for large n. Note that $\alpha(q) \leq \frac{q\ell k + \frac{1}{2}k(k-1)}{h_p+k} + 1$ and

$k \geq h_p$ so $\frac{\alpha(q)}{k} \leq \frac{q\ell+\frac{1}{2}k+1}{k} \leq \frac{2r_p+1}{h_p} + \frac{1}{2} \to \frac{1}{2}$. Set $\beta(q) = \alpha(q)h_p - q\ell(k-\alpha(q)) - \frac{1}{2}(k-\alpha(q))(k-\alpha(q)-1)$ so that $0 \leq \beta(q) < h_p + q\ell + k + \frac{1}{2}$. Then $\beta(q)$ roughly corresponds to the height in the column where applying T^{kh_p} switches between acting like $T^{(k-\alpha(q))h_p}$ and like $T^{(k-\alpha(q)-1)h_p}$. For $0 \leq j < \ell$ and $0 \leq i < h_p$, by Lemma 4.1 part (ii),

$$T^{kh_p} I_{p,i}^{[j+q\ell]} = T^{-j(k-\alpha(q))+\beta(q)} I_{p,i}^{[j+q\ell+k-\alpha(q)]}.$$

For $0 \leq i < h_p - \beta(q)$, we have

$$T^{-j(k-\alpha(q))+\beta(q)} I_{p,i}^{[j+\ell+k-\alpha(q)]} = T^{-j(k-\alpha(q))} I_{p,i+\beta(q)}^{[j+q\ell+k-\alpha(q)]}$$

and for $h_p - \beta(q) + q\ell - \alpha(q) + k \leq i < h_p$, we have

$$\begin{aligned} T^{-j(k-\alpha(q))+\beta(q)} I_{p,i}^{[j+\ell+k-\alpha(q)]} &= T^{-j(k-\alpha(q))+h_p} I_{p,i+\beta(q)-h_p}^{[j+q\ell+k-\alpha(q)]} \\ &= T^{-j(k-\alpha(q)+1)} I_{p,i+\beta(q)-h_p-q\ell-k+\alpha(q)}^{[j+q\ell+k-\alpha(q)+1]}. \end{aligned}$$

Hence for each q,

$$\begin{aligned} &\sum_{i=0}^{h_p-1} \Big| \sum_{j=0}^{\ell-1} \mu(T^{kh_p} I_{p,i}^{[j+q\ell]} \cap B) - \mu(I_{p,i}^{[j+q\ell]})\mu(B) \Big| \\ &\quad \leq \sum_{i=0}^{h_p-1} \Big| \sum_{j=0}^{\ell-1} \mu(T^{-j(k-\alpha(q))} I_{p,i}^{[j+q\ell+k-\alpha(q)]} \cap B) - \mu(I_{p,i}^{[j+q\ell]})\mu(B) \Big| \\ &\qquad + \sum_{i=0}^{h_p-1} \Big| \sum_{j=0}^{\ell-1} \mu(T^{-j(k-\alpha(q)+1)} I_{p,i}^{[j+q\ell+k-\alpha(q)+1]} \cap B) - \mu(I_{p,i}^{[j+q\ell]})\mu(B) \Big| \\ &\qquad + (q\ell + k - \alpha(q)) \frac{\ell}{r_p} \mu(I_{p,0}) \end{aligned}$$

and since $q\ell + k - \alpha(q) \leq r_p$,

$$\sum_{q=0}^{Q-1} (q\ell + k - \alpha(q)) \frac{\ell}{r_p} \mu(I_{p,0}) \leq \frac{Q\ell r_p}{r_p h_p} \leq \frac{r_p}{h_p} \to 0.$$

By Lemma 4.3,

$$\begin{aligned} &\sum_{q=0}^{Q-1} \sum_{i=0}^{h_p-1} \Big| \sum_{j=0}^{\ell-1} \mu(T^{-j(k-\alpha(q))} I_{p,i}^{[j+q\ell+k-\alpha(q)]} \cap B) - \mu(I_{p,i}^{[j+q\ell+k-\alpha(q)]})\mu(B) \Big| \\ &\quad \leq \sum_{q=0}^{Q-1} \frac{\ell}{r_p} \int \Big| \frac{1}{\ell} \sum_{j=0}^{\ell-1} \chi_B \circ T^{-j(k-\alpha(q))} - \mu(B) \Big| d\mu + \sum_{q=0}^{Q-1} \frac{\ell^2(k-\alpha(q))}{r_p h_p}. \end{aligned}$$

Note that $Q\ell \leq \frac{r_p - k}{\ell}$ by hypothesis so we have that $\sum_{q=0}^{Q-1} \frac{\ell^2(k-\alpha(q))}{r_p h_p} \leq \frac{\ell^2 Qk}{r_p h_p} \leq \frac{2\ell k}{h_p} \to 0$ by hypothesis, and, applying the above also to the $i \geq h_p - \beta(q) + q\ell + k - \alpha(q)$ case,

$$\sum_{i=0}^{h_p-1} \Big| \sum_{j=0}^{r_p-k-1} \mu(T^{kh_p} I_{p,i}^{[j]} \cap B) - \mu(I_{p,i}^{[j]})\mu(B) \Big|$$

$$\leq \frac{1}{Q}\sum_{q=0}^{Q-1}\int \Big|\frac{1}{\ell}\sum_{j=0}^{\ell-1}\chi_B\circ T^{-j(k-\alpha(q))}-\mu(B)\Big|d\mu$$
$$+\frac{1}{Q}\sum_{q=0}^{Q-1}\int \Big|\frac{1}{\ell}\sum_{j=0}^{\ell-1}\chi_B\circ T^{-j(k-\alpha(q)+1)}-\mu(B)\Big|d\mu+\gamma$$

where $\gamma=\gamma_n\to 0$ is the sum of all the terms above that tend to zero. Thus the above quantity goes to zero by the final hypothesis. Applying the entire argument again to the $k+1$ case implies that $\{t_n\}$ is mixing. □

Remark. *Note that the $\alpha(q)$ range from 0 to $R\approx\frac{h_p}{Q}$ (all quantities here of course depend on n) and that each particular value is taken on with approximately the same density over j. This is due to the fact that $\alpha(q)$ is roughly $\frac{jk}{h_p}$.*

Having obtained our above results that mixing on sequences will follow from convergence to zero of ergodic type averages of T^k where k is allowed to move with n (the number of terms being averaged), we now show that such averages do in fact converge to zero.

We shall need the Block Lemma, our next statement, which tells us that an ergodic type average of powers of T^k will be dominated by an ergodic type average of powers of T^{kq} with proportionally fewer terms.

Lemma 4.6. [Ada98] **(Block Lemma)** *Let T be a measure-preserving transformation and $B\in\mathcal{B}$. Then for any $R,L,q\in\mathbb{N}$,*

$$\int\Big|\frac{1}{R}\sum_{j=0}^{R-1}\chi_B\circ T^{-j}-\mu(B)\Big|d\mu\leq\int\Big|\frac{1}{L}\sum_{j=0}^{L-1}\chi_B\circ T^{-jq}-\mu(B)\Big|d\mu+\frac{qL}{R}.$$

Proof. The main idea is to split the sum into blocks of size Lp and then each Lp block into L blocks and use the measure-preserving property to combine terms. Details are left to the reader. □

The case when $k(n)$ is bounded follows directly from the total ergodicity of T (which follows since T has a mixing sequence). Our next proposition is that if $\frac{k(n)}{n}$ is bounded then the ergodic type averages of n consecutive powers of $k(n)$ converges to zero.

Proposition 4.7. *Let T be a staircase transformation. Then for any $\{k_n\}$ such that $\limsup_n\frac{k_n}{n}<\infty$ and any $B\in\mathcal{B}$,*

$$\lim_{n\to\infty}\int\Big|\frac{1}{n}\sum_{j=0}^{n-1}\chi_B\circ T^{jk_n}-\mu(B)\Big|d\mu=0.$$

Proof. Here we use the Block Lemma to multiply k_n by a number q_n so that $k_nq_n\approx h_p$ for some p. Then the mixing behavior of h_n will yield the convergence of the average of (any number of terms of) consecutive powers of $T^{k_nq_n}$ which by the Block Lemma dominates the average of powers of T^{k_n}. We will need that $\frac{k_n}{n}$ is bounded to ensure that the terms in the sum dropped by the Block Lemma (the $\frac{pL}{R}$ term) is small.

Formally, let $B\in\mathcal{B}$. Let $\{k_n\}$ be an arbitrary (not necessarily increasing) sequence. Set p, q and x such that $h_p\leq k<h_{p+1}\leq kq<2h_{p+1}$ and $xh_p\leq k<$

$(x+1)h_p$ (again dropping the n). Fix $\epsilon > 0$. Choose L such that $\frac{1}{L} < \epsilon$ (L does not depend on n). Note that $\frac{q}{n} = \frac{qk^2}{nk^2} \leq \frac{2h_{p+1}}{h_p^2}\frac{k}{n} \to 0$ since $\frac{h_{p+1}}{h_p^2} \approx \frac{r_p}{h_p} \to 0$ as T is finite measure-preserving. So for large n, $\frac{qL}{n} < \epsilon$.

By the Block Lemma,

$$\int \Big|\frac{1}{n}\sum_{j=0}^{n-1} \chi_B \circ T^{-jk} - \mu(B)\Big| d\mu \leq \int \Big|\frac{1}{L}\sum_{d=0}^{L-1} \chi_B \circ T^{-dkq} - \mu(B)\Big| d\mu + \frac{qL}{n}.$$

The result then follows by applying the following Lemma with $t = kq$ and then letting $\epsilon \to 0$: □

Lemma 4.8. *Let T be a staircase transformation with heights $\{h_n\}$ and let $\{t_n\}$ and $\{p_n\}$ be sequences such that $h_{p_n} \leq t_n < 2h_{p_n}$. Then for any $\epsilon > 0$ there exists L such that for all sufficiently large n,*

$$\int \Big|\frac{1}{L}\sum_{d=0}^{L-1} \chi_B \circ T^{-dt_n} - \mu(B)\Big| d\mu < \epsilon.$$

Proof. By Proposition 4.2, $\{h_n\}$ is mixing with respect to T hence T is weak mixing so totally ergodic. For $0 < d < L$, $h_{p+1} \leq dt < 2Lh_{p+1}$. By Proposition 4.4, the sequence $\{dt\}$ is then mixing with respect to T since $\frac{r_{p+1}(2L)}{h_{p+1}} \to 0$ and since T is totally ergodic. The result now follows from the next Lemma. □

Lemma 4.9. *Let T be a staircase transformation and $\{t_n\}$ a sequence such that for any fixed d the sequence $\{dt_n\}$ is mixing with respect to T. Then for any $\epsilon > 0$ there exists L such that for all sufficiently large n,*

$$\int \Big|\frac{1}{n}\sum_{j=0}^{n-1} \chi_B \circ T^{-jt_n} - \mu(B)\Big| d\mu < \epsilon.$$

Proof. For each d, let $N_d \in \mathbb{N}$ such that for all $n \geq N_d$, $\big|\mu(T^{dt}B \cap B) - \mu(B)\mu(B)\big| < \epsilon$. Pick L such that $\frac{1}{L} < \epsilon$. Pick N such that $\frac{L}{N} < \epsilon$. Then for all $n \geq \max_{0<d<L} N_d$, apply the Block Lemma and then Hölder's Inequality so

$$\begin{aligned}
&\int \Big|\frac{1}{n}\sum_{j=0}^{n-1} \chi_B \circ T^{-jt} - \mu(B)\Big| d\mu \\
&\quad\leq \int \Big|\frac{1}{L}\sum_{d=0}^{L-1} \chi_B \circ T^{-dt} - \mu(B)\Big| d\mu + \frac{L}{n} \\
&\quad\leq \Big[\int \Big|\frac{1}{L}\sum_{d=0}^{L} \chi_B \circ T^{-dt} - \mu(B)\Big|^2 d\mu\Big]^{\frac{1}{2}} + \epsilon \\
&\quad= \Big[\frac{1}{L}\sum_{d=-L+1}^{L-1} \frac{L-|d|}{L}\big(\mu(T^{dt}B \cap B) - \mu(B)\mu(B)\big)\Big]^{\frac{1}{2}} + \epsilon \\
&\quad< \Big[\frac{1}{L}\sum_{0<|d|<L} \frac{L-|d|}{L}\epsilon + \frac{1}{L}\mu(B)\Big]^{\frac{1}{2}} + 2\epsilon < \sqrt{2\epsilon} + 2\epsilon.
\end{aligned}$$

□

We now establish the final result needed to prove staircases are mixing: that the ergodic type average of powers of T^{k_n} will converge to zero regardless of how fast k_n grows. This property is of some independent interest and is referred to as power ergodicity:

Proposition 4.10. *Let T be a staircase transformation. Then T is power ergodic.*

Proof. For any k we find the largest p such that $k \approx xh_p$ for some positive integer x. The proof is by cases. In the case when $\frac{x}{r_p}$ is bounded above zero, we may use the Block Lemma to multiply k by $q = \frac{r_p}{x}$ so that $kq \approx h_{p+1}$. In the case when $\frac{xr_p}{h_p}$ is small, $\frac{x}{r_p}$ will be bounded so the averages of the powers of T^x will vanish. Then $\{k_n\}$ will be mixing (Proposition 4.4) so its average must converge to zero.

Having disposed of these cases, if $\frac{xr_p}{h_p}$ is bounded then we can pick $\ell(n)$ satisfying the conditions of Proposition 4.5 and so that $\frac{x}{\ell(n)}$ is bounded. Then Proposition 4.7 tells us the average of powers of T^x vanishes so by Proposition 4.5 $\{k(n)\}$ is mixing. The final case is when $\frac{xr_p}{h_p} \to \infty$. We again find a sequence $\ell(n)$ and break the average into blocks of size $\ell(n)$ but in this case the values of α we obtain (as in the statement of Proposition 4.5) will effectively take on all values. We then use that T is weak mixing to show that the average of averages of powers of $T^{x-\alpha}$ tends to zero and so again $\{k_n\}$ is mixing.

Now we make this precise. Let $\{k_n\}$ be an arbitrary sequence. Write $k_n = k = xh_p + m$ as usual, dropping the n and picking p uniquely so that $h_p \leq k < h_{p+1}$ and then x and m accordingly.

Case 1a: $\liminf_n \frac{x}{r_p} = \delta > 0$. Pick a large fixed L and let $z = \lfloor \frac{h_{p+1}}{k} \rfloor$. Then by the Block Lemma,

$$\int \Big|\frac{1}{n}\sum_{j=0}^{n-1} \chi_B \circ T^{jk} - \mu(B)\Big| d\mu \leq \int \Big|\frac{1}{L}\sum_{\ell=0}^{L-1} \chi_B \circ T^{\ell z} - \mu(B)\Big| d\mu + \frac{L}{n}\frac{z}{k}$$

and $\frac{h_{p+1}}{k} \leq \frac{r_p h_p}{xh_p} = \frac{r_p}{x} \leq \frac{1}{\delta}$ so since L is fixed the rightmost term tends to zero. By Lemma 4.8 with $t = z$ the above can be arbitrarily small by taking L sufficiently large.

Case 1b: $\lim_n \frac{xr_p}{h_p} = 0$. Since $\frac{x}{r_p} \leq 2$ (recall that $\limsup \frac{x}{r_p} \leq 1$ by the construction of the unique x and p), by Proposition 4.7 (using $r_p - x$ in place of n in the proposition) we have that $\int \Big|\frac{1}{r_p}\sum_{j=0}^{r_p-x-1} \chi_B \circ T^{-jx} - \mu(B)\Big| d\mu \to 0$ (and also for $x+1$ in place of x). Then by Proposition 4.4 (using the hypothesis for this case) we have that $\{k_n\}$ is mixing. For any fixed ℓ the same argument shows that $\{\ell k_n\}$ is a mixing sequence. The result follows from Lemma 4.8.

Case 2: $\limsup_n \frac{xr_p}{h_p} < \infty$ (and not Cases 1a or 1b). Call Δ the limit supremum and δ the limit infimum of $\frac{xr_p}{h_p}$ (so $0 < \delta \leq \Delta < \infty$, the $\delta = 0$ case being covered by 1b and the $\Delta = \infty$ to be case 3). Pick $\epsilon = \epsilon_n \to 0$ such that $\epsilon\frac{h_p}{r_p} \to \infty$ (possible since T is finite measure-preserving) and such that $\epsilon\frac{r_p}{x} \geq \alpha > 0$ for some $\alpha > 0$ (requires that $\frac{x}{r_p} \to 0$ which we may assume to be the case by dropping to a subsequence of the n and noting that the subsequence of n where this does not tend to zero is covered by case 1 above; specifically fix $\epsilon > 0$ and split the sequence into the part where $\frac{r_p}{h_p} < \epsilon$ and where it is greater, handle each separately and then take $\epsilon \to 0$). Set $\ell = \epsilon\frac{h_p}{x}$ (modify ϵ slightly so that ℓ is an integer). Then $\ell = \ell_n \to \infty$ and

$\frac{\ell}{r_p} = \epsilon \frac{h_p}{x} r_p \leq \epsilon \frac{1}{\delta} \to 0$ and $\frac{\ell x}{h_p} = \epsilon \to 0$. Note that $\frac{x}{\ell} \leq \frac{1}{\epsilon}\frac{x}{r_p}\Delta \leq \frac{\Delta}{\alpha} < \infty$. Then by Proposition 4.7, $\int \left|\frac{1}{\ell}\sum_{j=0}^{\ell-1} \chi_B \circ T^{j(x+2-\alpha(q))} - \mu(B)\right| d\mu \to 0$ for any $\alpha(q)$ satisfying the requirements of Proposition 4.5. Moreover, the convergence is uniform over any choice of the $\alpha(q)$ since Proposition 4.5 applies to any choice of sequence of $\alpha(q)$ which in turn gives that $\{k_n\}$ is mixing. Clearly the same argument works for a constant multiple $\{dk_n\}$ so by Lemma 4.9 the result follows.

Case 3: $\limsup_n \frac{xr_p}{h_p} = \infty$ (and not Cases 1a, 1b nor 2). Since $\frac{x}{h_p} \leq \frac{r_p}{h_p} \to 0$ there exists $\ell = \ell_n \to \infty$ such that $\frac{x\ell}{h_p} \to 0$. Then $\frac{\ell}{r_p} = \frac{h_p}{xr_p}\frac{\ell x}{h_p} \to 0$. Set $Q = Q_n = \lceil \frac{r_p - x}{\ell} \rceil \to \infty$ (recall we have eliminated case 1b and may drop to a subsequence since the subsequence along which this is bounded will be covered by case 1b). Let $\alpha(q)$ be the sequence of values required in Proposition 4.5. By the Block Lemma,

$$\frac{1}{Q}\sum_{q=0}^{Q-1} \int \Big|\frac{1}{\ell}\sum_{j=0}^{\ell-1} \chi_B \circ T^{j(x+2-\alpha(q))} - \mu(B)\Big| d\mu$$
$$\leq \frac{1}{Q}\sum_{q=0}^{Q-1} \int \Big|\frac{1}{L}\sum_{j=0}^{L-1} \chi_B \circ T^{j(x+2-\alpha(q))} - \mu(B)\Big| d\mu + \frac{L}{\ell}.$$

Now $\alpha(q)$ takes on all integer values from 0 to R_n where $R_n \to \infty$ since $Q = Q_n \to \infty$, and takes on each particular value with approximately the same density (see Remark 4). Hence for each fixed $d \neq 0$ there is some Z_d such that $\frac{1}{Q}\sum_{q,r=0}^{Q-1} \left|\mu(T^{d(\alpha(q)-\alpha(r))}B \cap B) - \mu(B)\mu(B)\right| < \epsilon$ for any $Q \geq Z_d$ (since T is weakly mixing we have that T is mixing on a sequence of density one and likewise T^d is weak mixing so the claim follows).

By the Hölder Inequality and the measure-preserving property (repeatedly),

$$\Big(\frac{1}{Q}\sum_{q=0}^{Q-1} \int \Big|\frac{1}{L}\sum_{j=0}^{L-1} \chi_B \circ T^{j(x+2-\alpha(q))} - \mu(B)\Big| d\mu\Big)^4$$
$$\leq \Big(\frac{1}{Q}\sum_{q=0}^{Q-1} \frac{1}{L^2}\sum_{j,t=0}^{L-1} \mu(T^{(j-t)(x+2-\alpha(q))}B \cap B) - \mu(B)\mu(B)\Big)^2$$
$$\leq \Big(\frac{1}{L^2}\sum_{j,t=0}^{L-1} \int \Big|\frac{1}{Q}\sum_{q=0}^{Q-1} \chi_B \circ T^{-(j-t)\alpha(q)} - \mu(B)\Big| \circ T^{(j-t)(x+2)} d\mu\Big)^2$$
$$\leq \frac{1}{L^2}\sum_{j,t=0}^{L-1} \frac{1}{Q^2}\sum_{q,r=0}^{Q-1} \mu(T^{(j-t)(\alpha(q)-\alpha(r))}B \cap B) - \mu(B)\mu(B).$$

Hold L fixed and pick n such that $Q \geq Z_d$ for all $-L \leq d \leq L$ (and $d \neq 0$). Then the above is less than $\epsilon + \frac{1}{L}$ which tends to zero. Then by Proposition 4.5 we have that $\{k_n\}$ is mixing. In fact, any constant multiple of $\{k_n\}$ is as well so the result follows by Lemma 4.9. □

We are now ready to prove that staircases are mixing. Using Proposition 4.5 this reduces to showing that ergodic type averages tend to zero and the above tells us this is the case.

Proof. (of Theorem 1) Let $\{t_n\}$ be a sequence. Let $\{r_n\}$ be the cut sequence and $\{h_n\}$ the height sequence for T and set $p(n)$ and $k(n)$ as usual. Pick a sequence $\{\ell_n\}$

such that $\ell_n \to \infty$, $\frac{\ell_n k(n)}{h_{p(n)}} \to 0$ and $\frac{\ell_n}{r_{p(n)}} \to 0$ (possible since $\frac{k(n)}{h_{p(n)}} \leq \frac{r_{p(n)}}{h_{p(n)}} \to 0$). Since T is power ergodic (Proposition 4.10), for any $B \in \mathcal{B}$, $\int \big| \frac{1}{\ell_n} \sum_{j=0}^{\ell_n - 1} \chi_B \circ T^{-j(k(n)+2-\alpha(n,q))} - \mu(B) \big| d\mu \to 0$ uniformly over appropriate $\alpha(n,q)$. Then by Proposition 4.5, $\{t_n\}$ is mixing with respect to T. □

5. General Rank-One Transformations

We now establish the results we obtained on staircases for general rank-one transformations. The methods follow the same strategy as above but with many additional technical complications due to the fact that the spacer sequence is now arbitrary and no longer arithmetical so tricks such as the Block Lemma are no longer straightforward. The propositions below are direct analogues of those above. The reader is encouraged to become familiar with the concepts in the case of staircase transformations before reading the details of the general case.

5.1. Dynamical Sequences. Dynamical sequences are the natural representation of spacer sequences for rank-one transformations (see Section 3).

Definition 5.1. *A* **dynamical sequence** $\{s_{n,j}\}_{\{r_n\}}$ *is a doubly-indexed collection of integers* $s_{n,j}$ *for* $n \in \mathbb{N}$ *and* $j \in \mathbb{Z}_{r_n}$ *where* $\{r_n\}$ *is a given sequence, called the* **index sequence***, which must have the property that* $\lim_{n\to\infty} r_n = \infty$*. The integer* $s_{n,j}$ *is the* j^{th} **element** *of the dynamical sequence at the* n^{th} *stage.*

5.2. Partial Sums of Dynamical Sequences.

Notation. *Let* $\{s_{n,j}\}_{\{r_n\}}$ *be a dynamical sequence and* $n \in \mathbb{N}$, $j \in \mathbb{Z}_{r_n}$, $k \in \mathbb{Z}_{r_n - j}$*. The* k^{th} *partial sum of the* j^{th} *element at the* n^{th} *stage is*

$$s_{n,j}^{(k)} = \sum_{z=0}^{k-1} s_{n,j+z}.$$

Definition 5.2. *Let* $\{s_{n,j}\}_{\{r_n\}}$ *be a dynamical sequence and* $k \in \mathbb{N}$*. The* k^{th} **partial sum dynamical sequence** *is the dynamical sequence* $\{s_{n,j}^{(k)}\}_{\{r_n - k\}}$ *(*n *"begins" at the smallest value such that* $r_n \geq k$*) whose elements are the* k^{th} *partial sums of* $\{s_{n,j}\}_{\{r_n\}}$*. Let* $\{k_n\}$ *be a sequence such that* $k_n < r_n$ *for all* n*. The* $\{k_n\}^{th}$ **partial sum dynamical sequence** *of* $\{s_{n,j}\}_{\{r_n\}}$ *is the dynamical sequence* $\{s_{n,j}^{(k_n)}\}_{\{r_n - k_n\}}$*.*

5.3. Increasing Dynamical Sequences.

Definition 5.3. *A dynamical sequence* $\{s_{n,j}\}_{\{r_n\}}$ *is* **increasing***, which will be written as* $s_{n,j} \to \infty$*, when for every fixed* $M \in \mathbb{N}$, $\lim_{n\to\infty} \frac{1}{r_n} \#\{j \in \mathbb{Z}_{r_n} : |s_{n,j}| < M\} = 0$ *(the symbol* $\#$ *denotes cardinality).*

Note: this property was referred to as a dynamical sequence being monotonic in [CS04].

5.4. Ergodic Dynamical Sequences.

Definition 5.4. *A dynamical sequence* $\{s_{n,j}\}_{\{r_n\}}$ *is an* **ergodic dynamical sequence** *(with respect to a transformation* T*) when for all* $B \in \mathcal{B}$*,*

$$\lim_{n\to\infty} \int \Big| \frac{1}{r_n} \sum_{j=0}^{r_n - 1} \chi_B \circ T^{-s_{n,j}} - \mu(B) \Big| d\mu = 0.$$

5.5. Mixing and Ergodic Dynamical Sequences. The following is a standard generalization of the Blum-Hanson theorem from sequences to dynamical sequences.

Theorem 2. *A transformation T is mixing if and only if every increasing dynamical sequence is ergodic with respect to T.*

Proof. Let T be a mixing transformation and $\{s_{n,j}\}_{\{r_n\}}$ an increasing dynamical sequence. For any $B \in \mathcal{B}$ and any $\epsilon > 0$ there exists $M \in \mathbb{N}$ such that for all $m \geq M$ or $m \leq -M$ we have $|\mu(T^m(B) \cap B) - \mu(B)\mu(B)| < \epsilon$. As $\{s_{n,j}\}_{\{r_n\}}$ is increasing, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, $\#\{(j,\ell) \in \mathbb{Z}_{r_n} \times \mathbb{Z}_{r_n} : |s_{n,j} - s_{n,\ell}| < M\} < \epsilon r_n^2$. Then

$$\begin{aligned}
\int \Big|\frac{1}{r_n}\sum_{j=0}^{r_n-1} \chi_B \circ T^{-s_{n,j}} - \mu(B)\Big|^2 d\mu \\
&= \frac{1}{r_n^2}\sum_{j,\ell=0}^{r_n-1} \mu(T^{s_{n,j}-s_{n,\ell}} B \cap B) - \mu(B)\mu(B) \\
&\leq \frac{1}{r_n^2}\sum_{|s_{n,j}-s_{n,\ell}|\geq M} \big|\mu(T^{s_{n,j}-s_{n,\ell}} B \cap B) - \mu(B)\mu(B)\big| + \frac{1}{r_n^2}\epsilon r_n^2 \mu(B) \\
&< \frac{1}{r_n^2}\sum_{|s_{n,j}-s_{n,\ell}|\geq M} \epsilon + \epsilon\mu(B) < \epsilon(1+\mu(B)).
\end{aligned}$$

Conversely, suppose T is not mixing. As mixing is equivalent to Rényi mixing there exists $B \in \mathcal{B}$, $\delta > 0$ and a sequence $\{t_m\}$ such that $\mu(T^{t_m} B \cap B) - \mu(B)\mu(B) \geq \delta$ for all m. Define $\{s_{n,j}\}_{\{r_n\}}$ by $r_n = n$ and $s_{n,j} = t_j$. Then

$$\begin{aligned}
\int \Big|\frac{1}{r_n}\sum_{j=0}^{r_n-1} \chi_B \circ T^{-s_{n,j}} - \mu(B)\Big| d\mu &\geq \Big|\frac{1}{n}\sum_{j=0}^{n-1}\int_B \chi_B \circ T^{-t_j} - \mu(B) d\mu\Big| \\
&= \Big|\frac{1}{n}\sum_{j=0}^{n-1} \mu(T^{t_j} B \cap B) - \mu(B)\mu(B)\Big| \geq \delta.
\end{aligned}$$

Since $\{t_m\}$ is strictly increasing, $\{s_{n,j}\}_{\{r_n\}}$ is increasing. □

5.6. Levels of Rank-One Transformations.

Lemma 5.1. *Let T be a rank-one transformation with levels $\{I_{n,i}\}$, heights $\{h_n\}$, and spacers $\{s_{n,j}\}_{\{r_n\}}$. Let $p \in \mathbb{N}$, $i \in \mathbb{Z}_{h_p}$, $j \in \mathbb{Z}_{r_p}$, $k \in \mathbb{Z}_{r_p - j}$, $t \in \mathbb{Z}_{h_p - i}$ and B a union of levels in C_p. Then the following hold:*

$$(i) \quad I_{p,i} = \bigcup_{j=0}^{r_p-1} I_{p,i}^{[j]};$$

$$(ii) \quad T^{kh_p + s_{p,j}^{(k)}}(I_{p,i}^{[j]}) = I_{p,i}^{[j+k]}; \text{ and}$$

$$(iii) \quad \mu(T^t(I_{p,i}^{[j]}) \cap B) = \frac{1}{r_p}\mu(T^t(I_{p,i}) \cap B).$$

Lemma 5.2. *For any $p \in \mathbb{N}$, any $\Lambda \subseteq \mathbb{Z}_{h_p}$, any $\Gamma \subseteq \mathbb{N}$, any B a union of levels in C_p and any maps $f : \Gamma \to \mathbb{Z}$ and $g : \Gamma \to \mathbb{Z}_{r_p}$,*

$$\sum_{i\in\Lambda} \Big|\sum_{j\in\Gamma} \mu(T^{f(j)}(I_{p,i}^{[g(j)]}) \cap B) - \mu(I_{p,i}^{[g(j)]})\mu(B)\Big|$$

$$\leq \int \Big|\frac{1}{r_p}\sum_{j\in\Gamma}\chi_B\circ T^{f(j)} - \mu(B)\Big|d\mu + \big(\sup_{j\in\Gamma} f(j)\big)\frac{1}{h_p}\frac{\#\Gamma}{r_p}.$$

5.7. Mixing Sequences.

Proposition 5.3. *Let T be a rank-one transformation with spacers $\{s_{n,j}\}_{\{r_n\}}$ and heights $\{h_n\}$ and let $k \in \mathbb{N}$. If $\{s^{(k)}_{n,j}\}_{\{r_n-k\}}$ is ergodic (with respect to T) then $\{kh_n\}$ is mixing (with respect to T).*

Proof. Let B be a union of levels in C_N for some fixed $N \in \mathbb{N}$. For any sets $J_n \subseteq \mathbb{Z}_{r_n-k}$, apply Lemmas 5.1 (i) then 5.1 (ii) and finally Lemma 5.2, for any $n \geq N$,

$$\sum_{i=0}^{h_n-1}\big|\mu(T^{kh_n}(I_{n,i})\cap B) - \mu(I_{n,i})\mu(B)\big|$$
$$\leq \sum_{i=0}^{h_n-1}\Big|\sum_{j\in\mathbb{Z}_{r_n-k}\setminus J_n}\mu(T^{kh_n}(I^{[j]}_{n,i})\cap B) - \mu(I^{[j]}_{n,i})\mu(B)\Big| + \frac{k}{r_n} + \frac{\#J_n}{r_n}$$
$$= \sum_{i=0}^{h_n-1}\Big|\sum_{j\in\mathbb{Z}_{r_n-k}\setminus J_n}\mu(T^{-s^{(k)}_{n,j}}(I^{[j+k]}_{n,i})\cap B) - \mu(I^{[j+k]}_{n,i})\mu(B)\Big| + \frac{k}{r_n} + \frac{\#J_n}{r_n}$$
$$\leq \int\Big|\frac{1}{r_n}\sum_{j=0}^{r_n-k}\chi_B\circ T^{-s^{(k)}_{n,j}} - \mu(B)\Big|d\mu + \frac{2k}{r_n} + 2\frac{\#J_n}{r_n} + \frac{1}{h_n}\sup_{j\in\mathbb{Z}_{r_n-k}\setminus J_n} s^{(k)}_{n,j}.$$

As $\{s^{(k)}_{n,j}\}_{\{r_n-k\}}$ is ergodic with respect to T, we need only show that there exists sets $J_n \subseteq \mathbb{Z}_{r_n-k}$ such that $\frac{\#J_n}{r_n} \to 0$ and $\frac{1}{h_n}\sup_{j\notin J_n} s^{(k)}_{n,j} \to 0$. Suppose not. Then there exists $\delta > 0$ such that $s^{(k)}_{n,j} \geq \delta h_n$ for at least δr_n values of j (for infinitely many n). But then at least $\frac{1}{k}\delta r_n$ values of j are such that $s_{n,j} \geq \frac{\delta}{k}h_n$ so $\mu(S_n) \geq \frac{\delta^2}{k^2}r_n h_n\mu(I_{n+1,0}) = \frac{\delta^2}{k^2}\mu(C_n)$ contradicting that T is defined on a finite measure space. □

Proposition 5.4. *Let T be a rank-one transformation with height sequence $\{h_n\}$ and spacer sequence $\{s_{n,j}\}_{\{r_n\}}$ and let $\{t_n\}$ be a sequence. Set $p(n)$ and $k(n)$ such that $h_{p(n)} \leq t_n < h_{p(n)+1}$ and $k(n)h_{p(n)} \leq t_n < (k(n)+1)h_{p(n)}$. If there exists $J_n \subseteq \mathbb{Z}_{r_{p(n)}-k(n)}$ such that $\frac{\#J_n}{r_{p(n)}} \to 0$ and $\frac{1}{h_{p(n)}}\sup_{j\notin J_n} s^{(k(n))}_{n,j} \to 0$ and for all $B \in \mathcal{B}$,*

$$\int\Big|\frac{1}{r_{p(n)}}\sum_{j=0}^{r_{p(n)}-k(n)-1}\chi_B\circ T^{-s^{(k(n))}_{p(n),j}} - \mu(B)\Big|d\mu \to 0$$

then $\{t_n\}$ is mixing with respect to T.

Proof. Write $t_n = kh_p + m$ (writing $k = k(n)$, $p = p(n)$ and $m = m(n)$ as before) and let A, B be unions of levels in C_N for some $N \in \mathbb{N}$. For sufficiently large n, as in Proposition 4.2,

$$\big|\mu(T^{t_n}A\cap B) - \mu(A)\mu(B)\big|$$
$$\leq \sum_{i=0}^{h_p-1}\big|\mu(T^{kh_p}I_{p,i}\cap B) - \mu(I_{p,i})\mu(B)\big|$$

$$+\sum_{i=0}^{h_p-1}\left|\mu(T^{(k+1)h_p}I_{p,i}\cap B)-\mu(I_{p,i})\mu(B)\right|$$

By Lemma 5.1 part (i),

$$\sum_{i=0}^{h_p-1}\left|\mu(T^{kh_p}I_{p,i}\cap B)-\mu(I_{p,i})\mu(B)\right|$$
$$\leq \sum_{i=0}^{h_p-1}\Big|\sum_{j=0}^{r_p-k-1}\mu(T^{kh_p}I_{p,i}^{[j]}\cap B)-\mu(I_{p,i}^{[j]})\mu(B)\Big|$$
$$+\sum_{i=0}^{h_p-1}\sum_{j=r_p-k}^{r_p-1}\left|\mu(T^{kh_p}I_{p,i}^{[j]}\cap B)-\mu(I_{p,i}^{[j]})\mu(B)\right|$$

and the second summand tends to zero as in Proposition 4.2.

By Lemma 5.1 part (ii) and Lemma 5.2,

$$\sum_{i=0}^{h_p-1}\Big|\sum_{j=0}^{r_p-k-1}\mu(T^{kh_p}I_{p,i}^{[j]}\cap B)-\mu(I_{p,i}^{[j]})\mu(B)\Big|$$
$$\leq \int\Big|\frac{1}{r_p}\sum_{j\notin J_n}\chi_B\circ T^{-s_{p,j}^{(k)}}-\mu(B)\Big|d\mu+\big(\sup_{j\notin J_n}s_{p,j}^{(k)}\big)\frac{1}{h_p}+\frac{\#J_n}{r_p}$$
$$\leq \int\Big|\frac{1}{r_p}\sum_{j=0}^{r_p-k-1}\chi_B\circ T^{-s_{p,j}^{(k)}}-\mu(B)\Big|d\mu+\big(\sup_{j\notin J_n}s_{p,j}^{(k)}\big)\frac{1}{h_p}+2\frac{\#J_n}{r_p}.$$

The same reasoning holds for $(k+1)$. □

Definition 5.5. *Fix T a rank-one transformation with spacer sequence $\{s_{n,j}\}_{\{r_n\}}$. Let Q,p,k be positive integers and $\{\Gamma_q\}_{q=0}^{Q-1}$ be a partition of $\mathbb{Z}_{r_p-k}$ (that is $\Gamma_q\subseteq\mathbb{Z}_{r_p-k}$ are disjoint and $\cup\Gamma_q=\mathbb{Z}_{r_p-k}$). We say $(Q,p,k,\{\Gamma_q\})$* **respects the spacing arrangement at** *k of T when $s_{p,j}^{(k)}\leq s_{p,j'}^{(k)}$ for all $j\in\Gamma_q$ and $j'\in\Gamma_{q'}$ whenever $q\leq q'$.*

Given sequences $\{p_n\}$, $\{k_n\}$ and $\{Q_n\}$, a sequence of partitions $\{\Gamma_{n,q}\}_{Q_n}$ that each individually respect the spacing arrangement of T at k_n and also have the property that $\frac{Q_n}{r_{p_n}}\to 0$ is said to **respect the spacing arrangement** *for T with* **block sizes** *$\{b_n\}$ when $\#\Gamma_{n,q}\geq b_n$ for a density one set of q.*

Proposition 5.5. *Let T be a rank-one transformation with spacers $\{s_{n,j}\}_{\{r_n\}}$ and heights $\{h_n\}$ and let $\{t_n\}$ be a sequence. Set $p(n)$ and $k(n)$ such that $h_{p(n)}\leq t_n< h_{p(n)+1}$ and $k(n)h_{p(n)}\leq t_n<(k(n)+1)h_{p(n)}$.*

If there exists a sequence $\{L_n\}$ such that $L_n\to\infty$ and $\frac{L_n}{h_{p_n}}\frac{1}{r_{p_n}}\sum_{j=0}^{r_{p_n}-1}s_{p_n,j}\to 0$ that also has the property that every sequence of partitions that respects the spacing arrangement of T with block sizes $\{\frac{h_{p_n}}{L_n}\}$ is **ergodic** *with respect to T, meaning*

$$\lim_{n\to\infty}\sum_{q=0}^{Q_n-1} int\Big|\frac{1}{r_{p(n)}}sum_{j\in\Gamma_{n,q}}\chi_B\circ T^{-s_{p(n),j}^{(k(n)+2-\alpha(n,q))}}-\mu(B)\Big|d\mu=0,$$

then $\{t_n\}$ is mixing with respect to T.

Proof. Let B be a union of levels in C_N for some $N \in \mathbb{N}$. Let $n \in \mathbb{N}$ such that $t_n \geq h_N$. Set $m = t_n - kh_p$ (writing $k = k(n)$ and $p = p(n)$ and so forth) so $m \in \mathbb{Z}_{h_p}$. Set $\epsilon = \epsilon_n = \frac{1}{L_n}$. Let $\Psi = \Psi_n : \mathbb{Z}_{r_p - k} \to \mathbb{Z}_{r_p - k}$ be a map such that $s^{(k)}_{p,\Psi(j)} \leq s^{(k)}_{p,\Psi(j+1)}$ for all $j \in \mathbb{Z}_{r_p-k-1}$. Set $\ell(0) = 0$ and $\alpha(-1) = 0$ and then proceed inductively to define $\ell(q+1) = \ell_n(q+1)$ and $\alpha(q) = \alpha_n(q)$ given $\ell(q)$ and $\alpha(q-1)$ as follows: choose $\ell(q+1)$ to be the smallest positive integer less than $r_p - k$ such that

$$s^{(k-\alpha(q-1)+1)}_{p,\Psi(\ell(q+1))} - s^{(k-\alpha(q-1))}_{p,\Psi(\ell(q))} \geq \epsilon h_p$$

if such an integer exists and choose $\ell(q+1) = r_p$ and set $Q_n = q+1$ if not. Note that

$$Q_n \leq \frac{s^{(k)}_{p,\Psi(r_p-k-1)}}{\epsilon h_p} \leq \frac{s^{(r_p)}_{p,0}}{\epsilon h_p} = \frac{r_p \mu(S_p)}{\epsilon \mu(C_p)}$$

so $\frac{Q_n}{r_p} \to 0$. Choose $\alpha(q)$ such that

i) $(\alpha(q)-1)h_p + s^{(\alpha(q)-1)}_{p,\Psi(\ell(q))+k-\alpha(q)+1} < s^{(k)}_{p,\Psi(\ell(q))}$; and

ii) $s^{(k)}_{p,\Psi(\ell(q))} \leq \alpha(q)h_p + s^{(\alpha(q))}_{p,\Psi(\ell(q))+k-\alpha(q)}$.

The $\alpha(q)$ are clearly nondecreasing over q. Since $\frac{L}{h_p}\frac{1}{r_p}\sum_{j=0}^{r_p-1} s_{p,j} \to 0$ for large n we have $\alpha(q+1) - \alpha(q) \leq 1$.

Set $\beta(q) = \alpha(q)h_p + s^{(\alpha(q))}_{p,\Psi(\ell(q))+k-\alpha(q)} - s^{(k)}_{p,\Psi(\ell(q))}$ and note that $0 \leq \beta(q) < h_p + s_{p,\Psi(\ell(q))+k-\alpha(q)}$. Set $\beta'(q) = \beta(q) - s_{p,\Psi(\ell(q))+k-\alpha(q)}$ and note that $\beta'(q) < h_p$. For all $q \in \mathbb{Z}_q$, define the sets

$$\Gamma_q = \Gamma_{n,q} = \left\{ j \in \mathbb{Z}_{r_p} : s^{(k)}_{p,\Psi(\ell(q))} \leq s^{(k)}_{p,j} < s^{(k)}_{p,\Psi(\ell(q+1))} \right\}$$

and the maps $\Psi(q) = \Psi(n,q) : \mathbb{Z}_{\ell(q+1)-\ell(q)} \to \Gamma_q$ by $\Psi(q)(j) = \Psi(j + \ell(q))$ for all $j \in \mathbb{Z}_{\ell(q+1)-\ell(q)}$. As in Proposition 4.5, it suffices to show that

$$\sum_{i=0}^{h_p-1} \Big| \sum_{j=0}^{r_p-1} \mu(T^{kh_p} I^{[j]}_{p,i} \cap B) - \mu(I^{[j]}_{p,i})\mu(B) \Big| \to 0$$

and likewise for $(k+1)$. Note that

$$\begin{aligned}
&\sum_{i=0}^{h_p-1} \Big| \sum_{j=0}^{r_p-k-1} \mu(T^{-s^{(k)}_{p,j}}(I^{[j+k]}_{p,i}) \cap B) - \mu(I^{[j+k]}_{p,i})\mu(B) \Big| \\
&\quad = \sum_{i=0}^{h_p-1} \Big| \sum_{q=0}^{Q-1} \sum_{j\in\Gamma_q} \mu(T^{-s^{(k)}_{p,j}}(I^{[j+k]}_{p,i}) \cap B) - \mu(I^{[j+k]}_{p,i})\mu(B) \Big|
\end{aligned}$$

Now for any $q \in \mathbb{Z}_q$, any $i \in \mathbb{Z}_{h_p}$ and any $j \in \Gamma_q$, using Lemma 5.1 (ii),

$$\begin{aligned}
T^{-s^{(k)}_{p,j}}(I^{[j+k]}_{p,i}) &= T^{-\left(s^{(k)}_{p,j} - s^{(k)}_{p,\Psi(\ell(q))}\right) - s^{(k)}_{p,\Psi(\ell(q))}}(I^{[j+k]}_{p,i}) \\
&= T^{-\left(s^{(k)}_{p,j} - s^{(k)}_{p,\Psi(\ell(q))}\right) - s^{(k)}_{p,\Psi(\ell(q))} + \alpha(q)h_p + s^{(\alpha(q))}_{p,j+k-\alpha(q)}}(I^{[j+k-\alpha(q)]}_{p,i}) \\
&= T^{-\left(s^{(k)}_{p,j} - s^{(k)}_{p,\Psi(\ell(q))}\right) + \beta(q)}(I^{[j+k-\alpha(q)]}_{p,i}).
\end{aligned}$$

If $i < h_p - \beta(q)$ then

$$T^{-s_{p,j}^{(k)}}(I_{p,i}^{[j+k]}) = T^{-\left(s_{p,j}^{(k-\alpha(q))} - s_{p,\Psi(\ell(q))}^{(k-\alpha(q))}\right)}(I_{p,i+\beta(q)}^{[j+k-\alpha(q)]}).$$

If $i \geq h_p - \beta'(q)$ then

$$\begin{aligned} T^{-s_{p,j}^{(k)}}(I_{p,i}^{[j+k]}) &= T^{-\left(s_{p,j}^{(k-\alpha(q))} - s_{p,\Psi(\ell(q))}^{(k-\alpha(q))}\right)+\beta(q)-h_p-s_{p,j+k-\alpha(q)}}(I_{p,i}^{[j+k-\alpha(q)+1]}) \\ &= T^{-\left(s_{p,j}^{(k-\alpha(q)+1)} - s_{p,\Psi(\ell(q))}^{(k-\alpha(q)+1)}\right)+\beta(q)-h_p-s_{p,\Psi(\ell(q))+k-\alpha(q)}}(I_{p,i}^{[j+k-\alpha(q)+1]}) \\ &= T^{-\left(s_{p,j}^{(k-\alpha(q)+1)} - s_{p,\Psi(\ell(q))}^{(k-\alpha(q)+1)}\right)}(I_{p,i+\beta'(q)-h_p}^{[j+k-\alpha(q)+1]}) \end{aligned}$$

as $i + \beta'(q) - h_p \geq 0$ because $i \geq h_p - \beta'(q)$ and $i + \beta'(q) - h_p < i < h_p$ because $\beta'(q) < h_p$.

If $h_p - \beta(q) \leq i < h_p - \beta'(q)$ then, as above,

$$\begin{aligned} T^{-s_{p,j}^{(k)}}(I_{p,i}^{[j+k]}) &= T^{-\left(s_{p,j}^{(k-\alpha(q))} - s_{p,\Psi(\ell(q))}^{(k-\alpha(q))}\right)+\beta(q)-h_p-s_{p,j+k-\alpha(q)}}(I_{p,i}^{[j+k-\alpha(q)+1]}) \\ &= T^{-\left(s_{p,j}^{(k-\alpha(q)+1)} - s_{p,\Psi(\ell(q))}^{(k-\alpha(q))}\right)}(I_{p,i+\beta(q)-h_p}^{[j+k-\alpha(q)+1]}). \end{aligned}$$

Applying the first case and Lemma 5.2, then that T is measure-preserving,

$$\begin{aligned} \sum_{q=0}^{Q_n-1} \sum_{i=0}^{h_p-\beta(q)} &\big| \sum_{j\in\Gamma_q} \mu(T^{-s_{p,j}^{(k)}}(I_{p,i}^{[j+k]}) \cap B) - \mu(I_{p,i}^{[j+k]})\mu(B)\big| \\ &\leq \sum_{q=0}^{Q_n-1} \int \big|\frac{1}{r_p}\sum_{j\in\Gamma_q} \chi_B \circ T^{-\left(s_{p,j}^{(k-\alpha(q))} - s_{p,\Psi(\ell(q))}^{(k-\alpha(q))}\right)} - \mu(B)\big| d\mu \\ &\quad + \sum_{q=0}^{Q_n-1} \big(\sup_{j\in\Gamma_q} s_{p,j}^{(k-\alpha(q))} - s_{p,\Psi(\ell(q))}^{(k-\alpha(q))}\big)\frac{1}{h_p}\frac{\#\Gamma_q}{r_p} \\ &\leq \sum_{q=0}^{Q_n-1} \int \big|\frac{1}{r_p}\sum_{j\in\Gamma_{n,q}} \chi_B \circ T^{-s_{p,j}^{(k-\alpha(q))}} - \mu(B)\big| d\mu + \epsilon. \end{aligned}$$

Similarly, for the second and third cases above (with $k+1$ replacing k). Combining these three cases and letting $n \to \infty$, we have the result. □

6. Mixing Theorem

Theorem 3. *For a rank-one transformation T, the following are equivalent:*

(*i*) *T is a mixing transformation;*

(*ii*) *T is "rank-one uniform mixing": for all $B \in \mathcal{B}$,*

$$\lim_{n\to\infty} \sup_{h_n \leq t < h_{n+1}} \sum_{i=0}^{h_n-1} \big|\mu(T^t I_{n,i} \cap B) - \mu(I_{n,i})\mu(B)\big| = 0; \text{ and}$$

(*iii*) *every sequence of partitions that respects the spacing arrangement of T with block sizes tending to infinity is ergodic with respect to T.*

Proof. (ii) implies (i) follows as in the proofs of our Propositions. (iii) implies (ii) follows as in Proposition 5.5. (i) implies (iii) follows since if not then for some sequence of partitions that respect the spacing arrangement of T there is some

$\{q(n)\}$ with $\#\Gamma_{n,q(n)} \to \infty$ (as $\frac{Q_n}{r_n} \to 0$) and the dynamical sequence $\{s_{n,j}^{(k(n))}\}_{\Gamma_{n,q(n)}}$ is not ergodic with respect to T so by Theorem 2, T cannot be mixing. $\square$

7. Polynomial Staircase Transformations

7.1. Polynomial Power Ergodicity The term **polynomial** shall mean polynomials with rational coefficients that map integers to integers.

Proposition 7.1. *Let T be a power ergodic transformation. Then for any sequence of polynomials $\{p_n\}$ of bounded degree and all $B \in \mathcal{B}$,*

$$\lim_{n\to\infty} \int \Big|\frac{1}{n}\sum_{j=0}^{n-1} \chi_B \circ T^{-p_n(j)} - \mu(B)\Big| d\mu = 0.$$

Lemma 7.2. van der Corput's Inequality *For any complex numbers a_n such that $|a_n| \leq 1$ and any $N, L \in \mathbb{N}$,*

$$\Big|\frac{1}{N}\sum_{n=0}^{N-1} a_n\Big|^2 \leq \frac{N+L}{N}\Big(\frac{1}{L} + 2Re\Big[\frac{1}{L}\sum_{\ell=1}^{L-1}\frac{L-\ell}{L}\frac{1}{N}\sum_{n=0}^{N-\ell-1} a_{n+\ell}a_n\Big]\Big).$$

Proof. (of Proposition 7.1). Induct on the degree D of the polynomials. Assume the condition holds for all $\{p_n\}$ of degree less than D. Note that the $D = 1$ case corresponds to power ergodicity. Let $\{p_n\}$ be a sequence of polynomials of degree D. Let $B \in \mathcal{B}$. Fix $\epsilon > 0$. Fix $L \in \mathbb{N}$ such that $\frac{1}{L} < \epsilon$. For $n \in \mathbb{N}$ and $0 < \ell < L$, set $P_{n,\ell}(j) = p_n(j+\ell) - p_n(j)$. Write $p_n(j) = \sum_{a=0}^{D} c_{n,a} j^a$ for $c_{n,a} \in \mathbb{Q}$. Then $P_{n,\ell}(j) = \sum_{b=0}^{D-1}\big(\sum_{a=b+1}^{D} c_{n,a}\binom{a}{b}\ell^{a-b}\big)j^b$ is a polynomial in j of degree $D-1$. By the inductive hypothesis, for each $0 < \ell < L$, there exists $N_\ell \in \mathbb{N}$ such that for all $n \geq N_\ell$,

$$\int \Big|\frac{1}{n}\sum_{j=0}^{n-1} \chi_B \circ T^{-P_{n,\ell}(j)} - \mu(B)\Big| d\mu < \epsilon.$$

Set $N_0 = \lceil \frac{L}{\epsilon} \rceil$ so $\frac{\ell}{n} < \epsilon$ for all $0 < \ell < L$ and $n \geq N_0$. For $n \geq \max_{0 \leq \ell < L} N_\ell$,

$$\begin{aligned}&\Big|\frac{1}{n}\sum_{j=0}^{n-\ell-1} \mu(T^{-p_n(j+\ell)+p_n(j)}B \cap B) - \mu(B)\mu(B)\Big| \\ &\leq \int \Big|\frac{1}{n}\sum_{j=0}^{n-1} \chi_B \circ T^{-P_{n,\ell}(j)} - \mu(B)\Big| d\mu + \frac{\ell}{n} < 2\epsilon.\end{aligned}$$

By Hölder's Inequality and van der Corput's Inequality,

$$\begin{aligned}&\int \Big|\frac{1}{n}\sum_{j=0}^{n-1} \chi_B \circ T^{-p(j)} - \mu(B)\Big| d\mu \\ &\leq \Big[\frac{n+L}{n}\Big(\frac{1}{L} + \frac{2}{L}\sum_{\ell=1}^{L-1}\frac{L-\ell}{Ln}\sum_{j=0}^{n-\ell-1}\big(\mu(T^{-p_n(j+\ell)+p_n(j)}B \cap B) - \mu(B)\mu(B)\big)\Big)\Big]^{\frac{1}{2}} \\ &< \Big[(1+\epsilon)\big(\epsilon + \frac{2}{L}\sum_{\ell=1}^{L-1}\frac{L-\ell}{L}2\epsilon\big)\Big]^{\frac{1}{2}} \leq \sqrt{5\epsilon(1+\epsilon)}.\end{aligned}$$

$\square$

7.2. Polynomial Staircase Transformations. Let $\{p_n\}$ be a sequence of polynomials with bounded degree and uniformly bounded coefficients. A rank-one transformation with spacer sequence $\{s_{n,j}\}_{\{r_n\}}$ given by $s_{n,j} = p_n(j)$ is a **polynomial staircase transformation** when the polynomials are such that for every $L \in \mathbb{N}$,

$$\limsup_{n\to\infty} \frac{1}{r_n} \#\{j \in \mathbb{Z}_{r_n} : L \quad \text{divides} \quad p_n(j+1) - p_n(j)\} < 1.$$

A **staircase transformation** is then a linear polynomial staircase transformation.

Theorem 4. *Polynomial staircase transformations are mixing.*

Proposition 7.3. [Furstenberg] *Let $\{p_n\}$ be a sequence of polynomials of bounded degree and uniformly bounded coefficients. Then a dynamical sequence $\{s_{n,j}\}_{\{r_n\}}$ given by $s_{n,j} = p_n(j)$ is ergodic with respect to any totally ergodic transformation.*

Proof. Since $\int \left|\frac{1}{r_n}\sum_{j=0}^{r_n-1} \chi_B \circ T^{-p_n(j)} - \mu(B)\right| d\mu = \int \left|\frac{1}{r_n}\sum_{j=0}^{r_n-1} z^{p_n(j)}\right| d\sigma_B(z)$ where σ_B is the spectral measure of B under T, the result follows from Weyl's theorem on the equidistribution of polynomial sequences (that $\frac{1}{n}\sum_{j=0}^{n-1} z^{p_n(j)} \to 0$ for all $z \in S^1 \setminus \mathbb{Q}$). □

Proposition 7.4. *Let T be a rank-one transformation with spacers $\{s_{n,j}\}_{\{r_n\}}$ such that for each $L \in \mathbb{N}, L \neq 1$, $\limsup_{n\to\infty} \frac{1}{r_n}\#\{j \in \mathbb{Z}_{r_n-1} : L$ divides $s_{n,j+1} - s_{n,j}\} < 1$. Then T is totally ergodic.*

Proof. Suppose T^L is not ergodic for some $L \in \mathbb{N}$, $L \neq 0, 1$. Then there exists $B \in \mathcal{B}, 0 < \mu(B) < 1$, such that $T^L B = B$. We may assume L is minimal so there exists $\delta > 0$ such that $\mu(T^\ell B \cap B) < (1-\delta)\mu(B)$ for all $0 < \ell < L$. Set $\rho_n : \mathbb{Z}_L \to [0,1]$ by $\rho_n(\ell) = \frac{1}{r_n}\#\{j \in \mathbb{Z}_{r_n} : s_{n,j} = \ell \mod L\}$. By hypothesis there exists $\delta_1 > 0$ such that $\#\{j \in \mathbb{Z}_{r_n} : L \text{ divides } s_{n,j+1} - s_{n,j}\} \geq \delta_1 r_n$ for all sufficiently large n. So for at least $\delta_1 r_n$ values of j, $s_{n,j+1} \neq s_{n,j} \mod L$. Hence $\rho_n(\ell) \leq 1 - \delta_1$ for all $0 \leq \ell < L$. Let $\{h_n\}$ be the height sequence for T. For any $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that for all $n \geq N$ there exists B_n a union of levels in C_n such that $\mu(B \triangle B_n) < \epsilon$ (here $\triangle$ denotes symmetric difference). Write $B_n = \bigcup_{i\in\Lambda_n} I_{n,i}$. Since L is fixed, $\frac{L}{h_n} \to 0$ so we may assume that $T^\ell B_n, 0 \leq \ell < L$, is a union of levels in C_n. Set w_n, v_n such that $h_n = w_n L + v_n$ for some $0 \leq v_n < L$. Set $B'_n = T^{v_n} B_n$. Then B'_n is a union of levels in C_n. Hence

$$\begin{aligned}
\mu(T^{w_n L} B \cap B) &< \mu(T^{h_n} B_n \cap T^{v_n} B_n) + 2\epsilon \\
&= \sum_{i\in\Lambda_n} \sum_{j=0}^{r_n-1} \mu(T^{h_n} I_{n,i}^{[j]} \cap B'_n) + 2\epsilon \\
&\leq \frac{1}{r_n} \sum_{j=0}^{r_n-1} \mu(T^{-s_{n,j}} B_n \cap B'_n) + 2(\epsilon + \frac{1}{r_n} + \mu(S_n)) \\
&= \sum_{\ell=0}^{L-1} \rho_n(\ell)\mu(T^{-\ell-v_n} B_n \cap B_n) + 2(\epsilon + \frac{1}{r_n} + \mu(S_n)) \\
&< \sum_{\ell=0}^{L-1} \rho_n(\ell)\mu(T^{-\ell} B \cap B) + 2(2\epsilon + \frac{1}{r_n} + \mu(S_n)) \\
&\leq (1-\delta_1)\mu(B) + \delta_1(1-\delta)\mu(B) + 2(2\epsilon + \frac{1}{r_n} + \mu(S_n))
\end{aligned}$$

since $\rho_n(\ell) \leq 1-\delta_1$ for $0 \leq \ell < L$ and $\mu(T^\ell B \cap B) \leq (1-\delta)\mu(B)$ for $0 < \ell < L$. Then $|\mu(T^{w_n L} B \cap B) < (1-\delta\delta_1)\mu(B) + 4\epsilon + \frac{2}{r_n} + 2\mu(S_n)$. But $T^{w_n L} B = B$ is then a contradiction. □

Proposition 7.5. *Let T be a rank-one transformation and $\{s_{n,j}\}_{\{r_n\}}$ its spacer sequence such that for each fixed k, the partial sum sequence $\{s_{n,j}^{(k)}\}_{\{r_n-k\}}$ is ergodic with respect to T. Then for any sequence $\{k_n\}$ such that $\limsup \frac{k_n}{n} < \infty$ and any $B \in \mathcal{B}$,*

$$\int \Big|\frac{1}{n}\sum_{j=0}^{n-1} \chi_B \circ T^{jk_n} - \mu(B)\Big| d\mu = 0.$$

Proof. The proof of Proposition 4.7 carries over directly to the nonstaircase case. □

Proof. (of Theorem 4). Let T be a polynomial staircase transformation. Then T is totally ergodic by Proposition 7.4. Let $\{p_n\}$ be the polynomials defining the spacer sequence $\{s_{n,j}\}_{\{r_n\}}$ of degree at most $D \in \mathbb{N}$ and let $\{c_{n,a}\}$ for $a \in \mathbb{Z}_{D+1}$ be the coefficients of the $\{p_n\}$. Then, for any $j \in \mathbb{Z}_{r_n}, k \in \mathbb{Z}_{r_n - j}$,

$$\begin{aligned} s_{n,j}^{(k)} &= \sum_{z=0}^{k-1} p_n(j+z) = \sum_{z=0}^{k-1}\sum_{a=0}^{D} c_{n,a}(j+z)^a = \sum_{z=0}^{k-1}\sum_{a=0}^{D}\sum_{b=0}^{a} c_{n,a}\binom{a}{b} j^b z^{a-b} \\ &= \sum_{b=0}^{D}\Big(\sum_{z=0}^{k-1}\sum_{a=b}^{D} c_{n,a}\binom{a}{b} z^{a-b}\Big) j^b = p_{n,k}(j) \end{aligned}$$

are polynomials of degree at most D in j with lead coefficients $kc_{n,D}$.

Assume now that T is power ergodic. T is then polynomial power ergodic by Proposition 7.1. For any $\{k_n\}$ and $\{\Gamma_{n,q}\}, \{\Psi_{n,q}\}, \{\alpha_{n,q}\}_{\{Q_n\}}$ as in Proposition 5.5, each $\{s_{n,\Psi_{n,q}(j)}^{(k_n-\alpha_{n,q})}\}_{\{\#\Gamma_{n,q}\}}$ is itself a polynomial sequence (details are left to the reader) of degree at most D . Since $\frac{Q_n}{r_n} \to 0$, $\#\Gamma_{n,q} \to \infty$ uniformly over a density one set of q, the averages over the sets $\Gamma_{n,q}$ then tend uniformly to zero by polynomial power ergodicity. Proposition 5.5 then gives the result. It remains only to show that T is power ergodic. □

Proposition 7.6. *Let T be a polynomial staircase transformation. Then T is power ergodic.*

Proof. For each fixed $k \in \mathbb{N}$ the $p_{n,k}$ are ergodic (Proposition 7.3) since the coefficients of $p_{n,k}$ are uniformly bounded when k is fixed (see the proof above for an expression for the coefficients). Then by Proposition 7.5 the case when $\limsup_n \frac{k_n}{n} < \infty$ is done. Write $k = k_n = xh_p + m$ as usual. Cases 1a and 1b from Proposition 4.10 carry over directly (using Proposition 7.5). Hence we may assume that $\liminf_n \frac{xr_p}{h_p} > 0$ and $\frac{x}{r_p} \to 0$. Pick ℓ_n as in Case 3 of Proposition 4.10 and Q_n and $\alpha(q)$ as well (note that $\limsup_n \frac{h_p}{xr_p} < \infty$ is enough for that construction).

Then, using the techniques of the proof of Proposition 7.1 (applying van der Corput's Lemma $d-1$ times where d is the degree of the polynomials defining the spacer sequence), for any $\Gamma_{n,q}$ an appropriate subset of $\{s_{p,j}^{(x)}\}_{\{r_p - x\}}$ and any appropriate $\alpha(q) = \alpha(n,q)$,

$$\int \Big|\frac{1}{\#\Gamma_{n,q}} \sum_{j\in\Gamma_{n,q}} \chi_B \circ T^{-s_{p,j}^{(x+2-\alpha(q))}} - \mu(B)\Big| d\mu$$

$$\leq \Big(\int \Big|\frac{1}{\ell_n} \sum_{j=0}^{\ell_n-1} \chi_B \circ T^{j(x+2-\alpha(q))z(q)} - \mu(B)\Big| d\mu\Big)^{\delta} + \epsilon$$

for some $\delta > 0$, $z(q) = z(n,q)$ integers and $\epsilon = \epsilon_n \to 0$. By the Block Lemma we may replace ℓ_n by some large fixed L (with error $\frac{L}{\ell_n} \to 0$).

Now $\alpha(q)$ take on all integer values in an interval of length arbitrarily large so as in the proof of Proposition 4.10 Case 3, using that T is weak mixing (so T^d is weak mixing for all fixed $d \neq 0$) we have that the quantity above tends to zero when averaged over q. The result then follows as in Proposition 4.10 Case 3 (details here have been left to the reader as everything follows from techniques previously used in the paper). □

8. Specific Examples

8.1. Criterion for Finite Measure on Rank-One Transformations

Proposition 8.1. *A rank-one transformation with spacer sequence $\{s_{n,j}\}_{\{r_n\}}$ and heights $\{h_n\}$ is defined on a finite measure space if and only if*

$$\sum_{n=0}^{\infty} \frac{\bar{s}_n}{h_n} < \infty \quad \Big(\textit{where} \quad \bar{s}_n = \frac{1}{r_n} \sum_{j=0}^{r_n-1} s_{n,j}\Big).$$

Proof. Let T, $\{s_{n,j}\}_{\{r_n\}}$, $\{\bar{s}_n\}$, and $\{h_n\}$ be as above and let (X,μ) be the space T is defined on. Let $\{C_n\}$ denote the columns of the construction as sets, $\{I_n\}$ the base levels of the columns, and $\{S_n\}$ the spacers added (so $S_n = C_{n+1} \setminus C_n$). We see that

$$\mu(S_n) = \sum_{j=0}^{r_n-1} s_{n,j}\mu(I_{n+1}) = \Big(\frac{1}{r_n} \sum_{j=0}^{r_n-1} s_{n,j}\Big)\mu(I_n) = \frac{\bar{s}_n}{h_n}\mu(C_n)$$

and so

$$\frac{\mu(C_{n+1})}{\mu(C_n)} = \frac{\mu(C_n) + \mu(S_n)}{\mu(C_n)} = 1 + \frac{\bar{s}_n}{h_n}.$$

Then,

$$\log\Big(\frac{\mu(X)}{\mu(C_0)}\Big) = \log\Big(\prod_{n=0}^{\infty} \frac{\mu(C_{n+1})}{\mu(C_n)}\Big) = \sum_{n=0}^{\infty} \log(1 + \frac{\bar{s}_n}{h_n}) \approx \sum_{n=0}^{\infty} \frac{\bar{s}_n}{h_n}$$

using the approximation $\log(1+\epsilon) \approx \epsilon$ for small ϵ.

Since $\mu(X) < \infty$ if and only if $\frac{\mu(X)}{\mu(C_0)} < \infty$, the result follows. □

8.2. Specific Examples of Mixing Transformations.

Theorem 5. *For $D \in \mathbb{N}$ and $\delta \in \mathbb{R}^+$, let $T_{D,\delta}$ be the rank-one transformation with spacer sequence $\{s_{n,j}\}_{\{r_n\}}$ given by $s_{n,j} = j^D$ and $r_n = \lfloor h_n^{\frac{1}{D+\delta}} \rfloor$ (where $\{h_n\}$ is the heights for $T_{D,\delta}$). Then $T_{D,\delta}$ is a mixing transformation.*

Proof. By Theorem 4, we need only show the transformations are defined on a finite measure space. Let $D \in \mathbb{N}$ and $\delta \in \mathbb{R}^+$. Let $\{s_{n,j}\}_{\{r_n\}}$ be the spacers and $\{h_n\}$ the heights for $T_{D,\delta}$. Now, $\sum_{j=0}^{r_n-1} s_{n,j} = \sum_{j=0}^{r_n-1} j^D \approx r_n^{D+1}$. Since $h_{n+1} = r_n h_n + \sum_{j=0}^{r_n-1} s_{n,j}$, we see that $h_n \geq \prod_{z=0}^{n-1} r_z \geq 2^n$. Then,

$$\sum_{n=0}^{\infty} \frac{\bar{s}_n}{h_n} \approx \sum_{n=0}^{\infty} \frac{1}{r_n h_n} r_n^{D+1} = \sum_{n=0}^{\infty} \frac{r_n^D}{h_n} \approx \sum_{n=0}^{\infty} h_n^{\frac{D}{D+\delta}-1} \leq \sum_{n=0}^{\infty} \left(2^{\frac{\delta}{D+\delta}}\right)^{-n} < \infty$$

by the convergence of geometric series. Proposition 8.1 completes the proof. □

8.3. Ornstein's Transformation. Ornstein's original construction of rank-one mixing transformations involved placing spacer levels randomly using a uniform distribution so that almost surely the resulting transformation is mixing. The uniform distribution can be equally well interpreted as meaning that the spacer sequence almost surely has the property that the averages in Theorem 3 tend uniformly to zero so that mixing for these transformations follows from our theorem. The reader is referred to [CS04] for details.

8.4. A Note on Restricted Growth. The restricted growth condition in [CS04] is equivalent to $\frac{r_n \bar{s}_n}{h_n} \to 0$ for polynomial staircase transformations (details are left to the reader) and to Adams' condition $\frac{r_n^2}{h_n} \to 0$ for staircase transformations. For $T_{D,\delta}$, we see that $\frac{r_n \bar{s}_n}{h_n} \approx \frac{r_n^{D+1}}{h_n} \approx h_n^{\frac{D+1}{D+\delta}-1} = h_n^{\frac{1-\delta}{D+\delta}}$ and so $T_{D,\delta}$ has restricted growth if and only if $\delta > 1$. Hence, the theorems of [CS04] and [Ada98] apply only to $T_{D,\delta}$ with $\delta > 1$.

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