The joint universality and the generalized strong-recurrence for L-functions

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D: Critical strip, $\mathrm{meas}\{A\}$: Lebesgue measure of A,
$\nu_T\{\dots\} := T^{-1}\mathrm{meas}\{\tau \in [0,T] : \dots\}$.
K: compact subsets of D with connected complement.

Thm. For almost all $\underline{\delta} \in \mathbb{R}$, for any compact subset K and for any $\varepsilon > 0$, we have

$$\liminf_{T\to\infty} \nu_T\left\{\sup_{s\in K}\left|L(s+i\tau,\chi) - L(s+i\underline{\delta}\tau,\chi)\right| < \varepsilon\right\} > 0.$$

When $\delta = \pm 1$ or δ is algebraic irrational, the above statement is true.
We hope that the above statement is true for all $\underline{\delta} \in \mathbb{R}$.

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Prop. [Bagchi]. The G. R. H. is true $\iff$ for any compact subset K and for any $\varepsilon > 0$, we have

$$\liminf_{T\to\infty} \nu_T\left\{\sup_{s\in K}\left|L(s+i\tau,\chi) - L(s,\chi)\right| < \varepsilon\right\} > 0.$$

1 Value-distribution of *L*-functions.

In the half-plane of absolute convergence $\sigma > 1$,

$$0 < |\zeta(s)| \le \zeta(\sigma), \qquad s = \sigma + it.$$

The study of the distribution of the values of the Riemann zeta function $\zeta(\sigma + it)$ for fixed σ and variable $t > 0$ was investigated by H. Bohr.

Thm. A [Bohr and Courant]. Let σ_0 be a fixed number in the range $1/2 < \sigma < 1$.
Then the values which $\zeta(s)$ takes on $\sigma = \sigma_0$, $t > 0$, are everywhere dense in the whole plane.

Thm. A was generelized by S. M. Voronin in 1972.

Thm. B [Voronin]. If $s_1, s_2, \ldots, s_m$ are distinct points lying in the strip $1/2 < \sigma < 1$, and $h > 0$ is an arbitrary fixed number then the sequence

$$\big(\zeta(s_1 + inh), \zeta(s_2 + inh), \ldots, \zeta(s_m + inh)\big), \quad n \in \mathbb{N}$$

is dense in $\mathbb{C}^m$. Moreover, the sequence

$$\big(\zeta(s_0 + inh), \zeta'(s_0 + inh), \ldots, \zeta^{(m-1)}(s_0 + inh)\big),$$

$n \in \mathbb{N}$ is dense in $\mathbb{C}^m$ for any fixed s_0 such that $1/2 < \Re(s_0) \leq 1$.

2 Universality

Recall $\zeta(s)$ and $L(s,\chi)$ defined by

$$\zeta(s) := \prod_p \Big(1 - \frac{1}{p^s}\Big)^{-1}, \quad L(s,\chi) := \prod_p \Big(1 - \frac{\chi(p)}{p^s}\Big)^{-1}.$$

$\mathrm{meas}\{A\}$: Lebesgue measure of the set A,
$\nu_T\{\ldots\} := T^{-1}\mathrm{meas}\{\tau \in [0,T] : \ldots\}$ where in place of dots some condition satisfied by τ is to be written.

$D := \{s \in \mathbb{C} : 1/2 < \Re(s) < 1\}$,
K and $K_1, \ldots, K_m$ $(m \geq 2)$: compact subsets of the critical strip D with connected complements.

Thm. C [Voronin]. Let $f(s)$ be a non-vanishing continuous function on K and analytic in the interior. Then for every $\varepsilon > 0$, we have

$$\liminf_{T\to\infty} \nu_T \Big\{ \sup_{s\in K} \big|\zeta(s+i\tau) - f(s)\big| < \varepsilon \Big\} > 0,$$

$$\nu_T\{\ldots\} := T^{-1}\mathrm{meas}\{\tau \in [0,T] :\ \ldots\}.$$

This theorem means that any non-vanishing analytic function can be uniformly approximated by $\zeta(s)$.

Moreover the set of approximating shifts has positive lower density. This property is called the universality.

Thm. D [Voronin, Gonek, Bagchi]. For $1 \leq l \leq m$, let $\chi_1 \bmod q_1$,. . ., $\chi_m \bmod q_m$ be pairwise non-equivalent Dirichlet characters, and $f_l(s)$ be a non-vanishing continuous on K_l and analytic in the interior of K_l. Then for every $\varepsilon > 0$, we have

$$\liminf_{T\to\infty} \nu_T \left\{ \sup_{1\leq l\leq m} \sup_{s\in K_l} \left| L(s+i\tau,\chi_l) - f_l(s) \right| < \varepsilon \right\} > 0.$$

This property is called the joint universality.

The joint universality means a collection of $L(s,\chi_l)$ of non-equivalent characters uniformly approximate simultaneously non-vanishing analytic functions.

3 Strong-recurrence and R. H.

Riemann hypothesis

$$\zeta(s) \neq 0 \quad \text{for} \quad \Re(s) > 1/2.$$

Generalized Riemann hypothesis

$$L(s, \chi) \neq 0 \quad \text{for} \quad \Re(s) > 1/2.$$

Denote by $N(\sigma, T)$ the number of the zeros of $\zeta(s)$ for which $\Re(s) > \sigma$ and $0 < \Im(s) \leq T$.
Bohr and Landau (1914) proved the next zero density theorem. For any fixed σ with $1/2 < \sigma < 1$, we have

$$N(\sigma, T) \leq CT^{4\sigma(1-\sigma)+\varepsilon}, \qquad \exists C > 0.$$

Thm. $\mathbf{C}^{\mathbf{R}}$. Let $f(s)$ be a non-vanishing continuous function on K and analytic in the interior of K. Then for every $\varepsilon > 0$, we have

$$\liminf_{T\to\infty} \nu_T\left\{\sup_{s\in K}\left|\zeta(s+i\tau) - \underline{f(s)}\right| < \varepsilon\right\} > 0,$$

$$\nu_T\{\ldots\} := T^{-1}\text{meas}\{\tau \in [0,T] :\ \ldots\}.$$

Remark. It is impossible to approximate uniformly functions having zeros by a Dirichlet L-function.

What happens when $\underline{f(s)}$ is replaced by $\underline{L(s,\chi)}$.

Thm. E [Bagchi]. The G. R. H. is true $\Longleftrightarrow$ for any compact subset K and for any $\varepsilon > 0$,

$$\liminf_{T\to\infty} \nu_T \left\{ \sup_{s\in K} \left| L(s+i\tau,\chi) - \underline{L(s,\chi)} \right| < \varepsilon \right\} > 0.$$

This property is called strong-recurrence (self-similarity).

Sketch of the proof of Thm. E.

R. H. is true $\Longrightarrow$ use Thm. C $\Longrightarrow$ L. H. S. > 0.

R. H. is not true $\Longrightarrow$ use the zero density theorem

$\Longrightarrow$ L. H. S. $= 0$.

$$N(\sigma,T) = O\bigl(T^{4\sigma(1-\sigma)}\bigr), \qquad 1/2 < \sigma < 1.$$

4 Main theorems

Thm. $\mathbf{E}^{\mathbf{R}}$. The G. R. H. is true $\iff$

for any compact subset K and for any $\varepsilon > 0$,

$$\liminf_{T\to\infty} \nu_T\left\{\sup_{s\in K}\left|\underline{L(s+i\tau,\chi) - L(s,\chi)}\right| < \varepsilon\right\} > 0.$$

Thm. 1. For almost all $\delta \in \mathbb{R}$, for any compact subset K and for any $\varepsilon > 0$, we have

$$\liminf_{T\to\infty} \nu_T\left\{\sup_{s\in K}\left|\underline{L(s+i\tau,\chi) - L(s+i\delta\tau,\chi)}\right| < \varepsilon\right\} > 0.$$

If we could take $\delta = 0$, we could obtain strong-recurrence.

Thm. 2. Let $\delta_1 = 1$, $f_l(s)$ be a non-vanishing continuous function on K_l and analytic in the interior. Then for almost all $\delta_2 \in \mathbb{R}$ and every $\varepsilon > 0$, we have

$$\liminf_{T\to\infty} \nu_T \Big\{ \sup_{1\le l\le 2} \sup_{s\in K_l} \big| L(s+i\delta_l\tau,\chi) - f_l(s) \big| < \varepsilon \Big\} > 0.$$

$$\nu_T\{\dots\} := T^{-1}\mathrm{meas}\{\tau \in [0,T] :\ \dots\}.$$

Proof of Theorem 1. By Theorem 2 and

$$\big| L(s+i\tau,\chi) - L(s+i\delta\tau,\chi) \big| \le$$
$$\big| L(s+i\tau,\chi) - 1 \big| + \big| L(s+i\delta\tau,\chi) - 1 \big| < 2\varepsilon,$$

we obtain Theorem 1.

Thm. 3. Let $1 = d_1, d_2, \ldots, d_m$ be algebraic real numbers and linearly independent over $\mathbb{Q}$, $d \in \mathbb{R} \setminus \{0\}$, and $f_l(s)$ be a non-vanishing continuous function on K_l and analytic in the interior of K_l. Then for every $\varepsilon > 0$,

$$\liminf_{T\to\infty} \nu_T \Big\{ \sup_{1\le l\le m} \sup_{s\in K_l} \big| L(s+idd_l\tau, \chi) - f_l(s) \big| < \varepsilon \Big\} > 0,$$

$$\nu_T\{\ldots\} := T^{-1}\mathrm{meas}\{\tau \in [0,T] :\ \ldots\}.$$

Remark. $1, dd_1, dd_2$ are not always linearly independent over $\mathbb{Q}$ when $1, d_1, d_2$ are linearly dependent over $\mathbb{Q}$. For instance, consider the case $d^{-1} = (\sqrt{2} + \sqrt{3})$.

5 Proof of Voronin's theorem

Thm. $\mathbf{A}^{\mathbf{R}}$. Let $f(s)$ be a non-vanishing continuous function on K and analytic in the interior. Then $\forall \varepsilon > 0$,

$$\liminf_{T\to\infty} \nu_T \left\{ \sup_{s\in K} \left| \zeta(s+i\tau) - f(s) \right| < \varepsilon \right\} > 0.$$

$H(D)$: the space of analytic on D functions equipped with the topology of uniform convergence on compacta. Define on $(H(D), \mathcal{B}(H(D)))$ the probability measures

$$P^T(A) := \nu_T\{L(s+i\tau,\chi) \in A\}, \qquad A \in \mathcal{B}(H(D)),$$

γ: unit circle on $\mathbb{C}$, $\Omega := \prod_p \gamma(p)$, where $\gamma(p) = \gamma$ for p. Ω is a compact topological Abelian groups.

m_H : the probability Haar measure on $(\Omega, \mathcal{B}(\Omega))$

We obtain the probability spaces $(\Omega, \mathcal{B}(\Omega), m_H)$.

We define the $H(D)$-valued random element

$$L(s, \chi \,|\, \omega) := \prod_p \Big(1 - \frac{\chi(p)\omega(p)}{p^s}\Big)^{-1}, \quad s \in D, \;\; \omega \in \Omega.$$

Let P stand for the distribution of $L(s, \chi \,|\, \omega)$, i.e.

$$P(A) := m_H(\omega \in \Omega : L(s, \chi \,|\, \omega) \in A), \quad A \in \mathcal{B}(H(D)).$$

> **Prop. A** [Bagchi]. The probability measure P^T converges weakly to P as $T \to \infty$.

Proof. $\{\log p_n\}_{n \in \mathbb{N}}$ is linearly independent over $\mathbb{Q}$.

Key. The unique factorization of prime numbers.

6 Proof of main theorems

Thm. $\mathbf{3}^{\mathbf{R}}$. Let $1 = d_1, d_2, \ldots, d_m \in \mathbb{R} \cap \overline{\mathbb{Q}}$ and lin. ind. over $\mathbb{Q}$, $d \in \mathbb{R} \setminus \{0\}$, and $f_l(s)$ be a non-vani. conti. func. on K_l and analytic in the interior. Then for $\forall \varepsilon > 0$,

$$\liminf_{T\to\infty} \nu_T \Big\{ \sup_{1\le l\le m} \sup_{s\in K_l} \big| L(s+idd_l\tau, \chi) - f_l(s) \big| < \varepsilon \Big\} > 0.$$

$H^m(D) := H(D) \times \cdots \times H(D)$, $\Omega^m := \Omega \times \cdots \times \Omega$.
Define on $(H^m(D), \mathcal{B}(H^m(D)))$ the probability measure
$\underline{P}_T(A) := \nu_T \big\{ \big(L(s+id_1\tau, \chi), \ldots, L(s+id_m\tau, \chi) \big) \in A \big\}$.

$\underline{m}_H$: the probability Haar measure on $(\Omega^m, \mathcal{B}(\Omega^m))$.
$\omega_l(p)$: the projection of $\omega_l \in \Omega$ to $\gamma(p)$.

Define on the probability space $(\Omega^m, \mathcal{B}(\Omega^m), \underline{m}_H)$ the random element $\big(L(s,\chi,\omega_1),\dots,L(s,\chi,\omega_m)\big)$, where

$$L(s,\chi,\omega_l) := \prod_p \left(1 - \frac{\chi(p)\omega_l(p)}{p^s}\right)^{-1}, \quad s \in D.$$

Let $\underline{P}_{\underline{L}}$ stand for the distribution of the random element $\underline{L}(s,\chi,\underline{\omega}) := \big(L(s,\chi,\omega_1),\dots,L(s,\chi,\omega_m)\big)$, i.e.
$\underline{P}_{\underline{L}}(A) := \underline{m}_H(\underline{\omega} \in \Omega^m : \underline{L}(s,\chi,\underline{\omega}) \in A)$.

Prop. 1. $\underline{P}_T$ converges weakly to $\underline{P}_{\underline{L}}$ as $T \to \infty$.

Proof. $\{\log p_n^{d_l}\}_{n\in\mathbb{N}}^{1\le l\le m}$ is linearly independent over $\mathbb{Q}$.

Key. Baker's theorem.

$\alpha_1^{\beta_1}\cdots\alpha_n^{\beta_n} \notin \overline{\mathbb{Q}}$ for any $\alpha_1,\dots,\alpha_n \in \overline{\mathbb{Q}}$, other than 0 or 1, and any $\beta_1,\dots,\beta_n \in \overline{\mathbb{Q}}$ with $1,\beta_1,\dots,\beta_n$ linearly independent $\mathbb{Q}$.

Thm. $\mathbf{2^R}$. Let $\delta_1 = 1$, $f_l(s)$ be a non-vanishing continuous function on K_l and analytic in the interior. Then for almost all $\delta_2 \in \mathbb{R}$ and every $\varepsilon > 0$, we have

$$\liminf_{T\to\infty} \nu_T \Big\{ \sup_{1\le l\le 2} \sup_{s\in K_l} \big| L(s+i\delta_l\tau,\chi) - f_l(s) \big| < \varepsilon \Big\} > 0,$$

$$\nu_T\{\ldots\} := T^{-1}\mathrm{meas}\{\tau \in [0,T] : \ldots\}.$$

Proof. For almost all $\delta_2 \in \mathbb{R}$, $\{\log p_n\} \cup \{\log p_n^{\delta_2}\}$ is linearly independent over $\mathbb{Q}$.

Key. Almost all numbers are transcendental numbers. The countable sum of null sets is also a null set.

7 Generalization

Def. A. $\tilde{\mathcal{S}}$ class, $\mathcal{L}(s) := \sum_{n=1}^{\infty} a(n)n^{-s}$.

(i) Ramanujan hypothesis. $a(n) = O(n^{\varepsilon})$, $\forall\varepsilon > 0$.

(ii) Analytic continuation. $\mathcal{L}(s)$ has an analytic conti. to $\sigma > \sigma_{\mathcal{L}}$ with $\sigma_{\mathcal{L}} < 1$ except for a pole at $s = 1$.

(iii) Finite order. For any fixed $\sigma > \sigma_{\mathcal{L}}$ and any $\varepsilon > 0$, $\mathcal{L}(s + it) = O(|t|^{\mu_{\mathcal{L}}+\varepsilon})$, $t \to \infty$.

(iv) Polynomial Euler product. $m, k \in \mathbb{N}$ and for p, $\alpha_j(p)$, $1 \leq j \leq k$, s.t. $\mathcal{L}(s) = \prod_p \prod_{r=1}^{n}(1 - \alpha_r(p)p^{-s})^{-1}$.

(v) Prime mean-square. $\pi(x)$: # primes satisfying $p \leq x$. $\lim_{x\to\infty} \pi(x)^{-1} \sum_{p\leq x} |a(p)|^2 = \kappa$.

Def. B. $\mathcal{S}$ class, $\mathcal{L}(s) := \sum_{n=1}^{\infty} a(n)n^{-s}$.

(1) Ramanujan hypothesis. $a(n) = O(n^{\varepsilon})$, $\forall \varepsilon > 0$.

(2) Analytic continuation. There exists a $k \in \mathbb{N} \cup \{0\}$ such that $(s-1)^k \mathcal{L}(s)$ is an entire function of finite order.

(3) Functional equation. $\mathcal{L}(s)$ satisfies $\lambda_{\mathcal{L}}(s) = \omega \overline{\lambda_{\mathcal{L}}(1-\bar{s})}$, $\lambda_{\mathcal{L}}(s) := \mathcal{L}(s) Q^s \prod_{j=1}^{f} \Gamma(\lambda_j s + \mu_j)$ with positive real numbers Q, λ_j and complex numbers μ_j, ω with $\Re(\mu_j) \geq 0$ and $|\omega| = 1$.

(4) Euler product. $\mathcal{L}(s)$ satisfies $\mathcal{L}(s) = \prod_p \mathcal{L}_p(s)$, $\mathcal{L}_p(s) = \exp\left(\sum_{j=1}^{\infty} b(p^j) p^{-js}\right)$ with suitable coefficients $b(p^j)$ satisfying $b(p^j) = O(p^{j\theta})$ for some $\theta < 1/2$.

$$d_{\mathcal{L}} := 2\sum_{j=1}^{f} \lambda_j, \qquad \sigma_* := 1 - \frac{1}{4(d_{\mathcal{L}}+3)}.$$

Prop. B [Kaczorowski and Perelli]. For every $\mathcal{L} \in \mathcal{S}$,

$$N_{\mathcal{L}}(\sigma, T) = O(T^{4(d_{\mathcal{L}}+3)(1-\sigma)+\varepsilon}), \quad 1/2 \leq \sigma \leq 1.$$

Thm. D [Steuding]. Let $\mathcal{L} \in \mathcal{S} \cap \tilde{\mathcal{S}}$. Then $\mathcal{L}(s)$ is non-vanishing in $\sigma_* < \Re(s) < 1 \iff$ for $\forall\varepsilon > 0$, $\sigma_* < \Re(\forall z) < 1$ and $0 < \forall r < \min\{\Re(z)-\sigma_*, 1-\Re(z)\}$,

$$\liminf_{T\to\infty} \nu_T \left\{ \sup_{|s-z|\leq r} \left|\mathcal{L}(s+i\tau) - \mathcal{L}(s)\right| < \varepsilon \right\} > 0.$$

Prop. 2. Let $\mathcal{L} \in \mathcal{S} \cap \tilde{\mathcal{S}}$ and $a_* := (1+\sigma_*)/2$. Then $\mathcal{L}(s)$ is non-vanishing in the half-plane $\sigma_* < \Re(s) < 1$ $\iff$ for any $\varepsilon > 0$ and any $0 < r_* < (1-\sigma_*)/2$,

$$\liminf_{U\to\infty} \nu_U^{\mu}\{\liminf_{T\to\infty} \nu_T^{\tau}\{\cdots\} > 0\} = 1,$$

$\cdots := \sup_{|s-a_*|\leq r_*} |\mathcal{L}(s+i\tau) - \mathcal{L}(s+i\underline{\mu})| < \varepsilon$.

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$$\liminf_{U\to\infty} \nu_U^{\mu}\{\liminf_{T\to\infty} \nu_T^{\tau}\{\cdots\} > 0\} = 1,$$

$$\cdots := \sup_{|s-a_*|\le r_*} |\mathcal{L}(s+i\tau) - \mathcal{L}(s+i\underline{\mu})| < \varepsilon.$$

Thm. 4. Let $\mathcal{L} \in \mathcal{S} \cap \tilde{\mathcal{S}}$ and $a_* := (1+\sigma_*)/2$. Then for any $\varepsilon > 0$ and any $0 < r_* < (1-\sigma_*)/2$, it holds that

$$\liminf_{U\to\infty} \nu_U^{\mu}\{\liminf_{T\to\infty} \nu_T^{\tau}\{\cdots\} > 0\} = 1,$$

$$\cdots := \sup_{|s-a_*|\le r_*} |\mathcal{L}(s+i\tau) - \mathcal{L}(s+i\underline{\mu\tau})| < \varepsilon.$$