A Stochastic Model Relating Rainfall Intensity to Raindrop Processes

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ABSTRACT

The temporal variability of rainfall and raindrop processes is examined at time scales ranging from less than 1 minute to 1 hour. Raindrop processes are represented in terms of drop arrival rate, mean diameter and coefficient of variation of drop diameter and modeled as time-varying stochastic processes. It is shown that rainfall rate and accumulated rainfall have simple and accurate representations in terms of raindrop processes. Using these results the temporal variability of rainfall rate is examined in terms of temporal variability of raindrop processes. It is shown that the temporal variability of rainfall rate varies systematically across a range of climatic settings and, more importantly, that these climatic contrasts in rainfall rate can be related to contrasting properties of raindrop processes. Two statistical models of rainfall rate and raindrop processes are examined in detail: a lognormal model with fixed parameters and a lognormal model with parameters that vary from storm to storm. Temporal correlation structure of rainfall rate exhibits qualitatively different behaviour under the two models. Lognormal models of rainfall rate are extended to models in which dependence on the averaging time interval is explicitly represented. Scale properties of rainfall rate are examined empirically for averaging time intervals ranging from 1 minute to 30 minutes. Empirical analyses are based on drop-size data from North Carolina, New Jersey, Oregon, Alaska, and the Marshall Islands.

1 Introduction

The temporal variability of rainfall is examined at time scales ranging from less than 1 minute to 1 hour. A representation of rainfall based on simplified dynamics of raindrop processes near the ground surface is used for analysis of rainfall variability. This representation provides the framework for both empirical studies of rainfall variability using drop-size data and development of analytical results that characterize temporal variability of rainfall. The importance of rainfall variability for land surface hydrology has been recognized for many years. Recent studies from research watersheds (see, for example, Goodrich and Woolhiser [1990]) have pointed to the need for better characterization of temporal variability of rainfall rate, especially at time scales shorter than 1 hour.

Rainfall has been routinely studied in terms of raindrop processes in fields such as soil ersoion, precipitation chemistry, cloud physics, radar hydrometeorology, and atmospheric remote sensing. Few hydrologic modeling studies of rainfall, however, have adopted a raindrop process framework (see Horton [1948] for a notable exception). In this paper the raindrop process framework provides a useful setting for analytical and empirical studies of rainfall variability over a range of time scales, as well as a direct modeling connection with related areas of study, especially remote sensing (see Smith and Krajewski [1993]) and land surface processes (see Smith and De Veaux [1992]).

The lognormal distribution, which has a long history in rainfall studies (see Gupta and Waymire [1991] and Kedem and Chiu [1987] for recent discussions), plays a prominent role in statistical analyses of rainfall and raindrop processes. The lognormal distribution provides a tractable model for empirical and theoretical analyses of accumulated rainfall, rainfall rate, and raindrop processes. In particular, the product representation of rainfall in terms of raindrop processes (Section 2) makes the lognormal distribution a natural choice for statistical analyses.

Several areas of research are drawn upon in this paper. The marked point process representation of raindrop processes developed in Smith [1993] provides a framework for model development. Important antecedents in stochastic process representations of rainfall include the works of Gupta and Waymire [1979], Waymire et al. [1984], Smith and Karr [1983], Rodriguez-Iturbe [1986] and Hosking and Stow [1987]. Related references on scaling properties of rainfall include the works by Rodriguez-Iturbe [1991], Rodriguez-Iturbe et al. [1989], Sharifi et al. [1990], Gupta and Waymire [1991, 1992], Lovejoy and Schertzer [1985], and Zawadzki [1987].

2 Rainfall and Raindrop Processes

In this section a model for raindrop processes is presented and results relating raindrop processes and rainfall are given. The model represents raindrop processes during a continuous period of rain, which we term a storm. Empirical analyses are based on drop-size data collected at sites in North Carolina, New Jersey, Oregon, Alaska and the Marshall Islands using the Illinois State Water Survey raindrop camera. Observations consist of raindrop counts and diameters on a 1-minute time resolution. Detailed descriptions of the data are given in Cataneo and Stout [1968], Jones [1992] and Smith [1993].

Let $\eta(t)$ denote the number of raindrops that fall in a 1 square meter surface area from the onset of rainfall until time t (in seconds). Denote the diameter of the ith drop by D_i (in mm). Rainfall accumulation (in mm) can be represented in terms of the number and diameters of raindrops as follows (assuming that raindrops are perfect spheres):

$$A(t) = [\frac{\pi}{6}10^{-6}] \sum_{i=1}^{\eta(t)} {D_i}^3 \quad (1)$$

Rainfall rate (in mm h^{-1}) is defined relative to an accumulation time interval Δt as follows:

$$R(t) = 3600[\frac{A(t) - A(t - \Delta t)}{\Delta t}] \tag{2}$$

Rainfall rate, as defined above, depends explicitly on a fixed time interval Δt (in the notation, we will suppress dependence of rainfall rate on Δt until section 4). For empirical analyses presented in section 3, Δt is less than 1 minute. The raindrop camera collects samples each minute from a 1 m^3 sample volume; at the beginning of the minute the camera is activated for a period of approximately 10 seconds, then it is inactive until the next minute.

The representation of rainfall in (1) and (2) is easily modified to account for a sample area different from 1 square meter by dividing the right-hand side of (1) (or (2)) by the sample area in m^2. The value of 1 square meter is chosen for consistency with the approximate observation scale of the raindrop camera and to reflect the approximate scale of hydrologic plot studies.

The model of Smith [1993] for drop-size distributions is used in analyzing the relationship between rainfall rate and raindrop processes. In the model $(\lambda(t), m(t), c(t))$ are stochastic processes with the following interpretation: $\lambda(t)$ is the mean arrival rate of raindrops (in drops $m^{-2}s^{-1}$) at time t; m(t) is the mean diameter (in mm) of raindrops at time t; and c(t) is the coefficient of variation of drop diameters at time t (i.e., the standard deviation of drop diameter divided by the mean drop diameter). The model for raindrop processes is specified by the assumptions that: 1) the drop arrival process is a Poisson process with randomly varying rate of occurrence $\lambda(t)$ and 2) drop diameters have a lognormal distribution with time-varying mean m(t) and time-varying coefficient of variation c(t) (the notation CV will be used throughout the paper for coefficient of variation).

The representation of $(\lambda(t), m(t), c(t))$ as stochastic processes that vary over the course of a storm reflects the natural variability of raindrop processes (see, for example, Waldvogel [1974]). From observations of raindrop counts and diameters, it is possible to reconstruct the raindrop processes $(\lambda(t), m(t), c(t))$ (or, in the language of stochastic processes, develop state estimators for the processes $(\lambda(t), m(t), c(t))$). Figure 1 illustrates temporal variability of raindrop processes for two storms that occurred in North Carolina in 1961. In Smith [1993], connections are drawn with other representations of raindrop processes, including the Marshall-Palmer model (Marshall and Palmer [1948] and Waldvogel's model [1974].

In the appendix it is shown that rainfall accumulation can be related to raindrop processes as follows:

$$A(t) = [\frac{\pi}{6}10^{-6}] \int_0^t \lambda(u) m(u)^3 (1 + c(u)^2)^3 du \tag{3}$$

It follows that rainfall rate can be represented in terms of raindrop processes by:

$$R(t) = \frac{[6\pi 10^{-4}]}{\Delta t} \int_{t-\Delta t}^t \lambda(u) m(u)^3 (1 + c(u)^2)^3 du \tag{4}$$

If raindrop processes are constant over the time interval $[t - \Delta t, t]$, then

$$R(t) = [6\pi 10^{-4}] \lambda(t) m(t)^3 (1 + c(t)^2)^3 \tag{5}$$

In practical terms, this representation can be used whenever raindrop processes are slowly varying relative to the time interval Δt.

The relationship between rainfall rate and raindrop processes given by (5) is quite accurate. Figure 2 compares model rainfall estimates (i.e., rainfall estimates derived from (5) using raindrop process observations) with observed rainfall rate values for the two North Carolina storms illustrated in figure 1. The storms have contrasting characteristics in terms of temporal variability of rainfall rate. For the first storm, rainfall rate varies from 1 mm h^{-1} to 10 mm h^{-1}; for the second storm, rainfall rate varies from 0.1 mm h^{-1} to 100 mm h^{-1}. For both cases model estimates of rainfall rate accurately reproduce the magnitude and variability of observed rainfall rate.

The results of figure 2 are representative of model performance. For 4,741 drop-size samples from North Carolina, the error in representing rainfall rate by (5) is less than 10% for more than 99.9% of the samples; the error is less than 2% for more than 99% of the samples. Model errors do not appear to be related to the magnitude of rainfall rate, drop arrival rate or mean diameter (figure 3). The range in model errors does increase with variability in drop diameter (figure 3). There is, however, little evidence for systematic overprediction or underprediction of rainfall rate as variability in drop diameter increases.

Table 1 summarizes moment properties of rainfall rate and raindrop processes for five sites (the notation E[ ] denotes expected value and CV denotes coefficient of variation). Mean rainfall rate ranges from 11.06 mm h^{-1} for the Marshall Islands to 1.28 mm h^{-1} for Alaska. These values represent mean rainfall conditioned on rain occurrence. The coefficient of variation of rainfall rate varies from 1.06 (Oregon) to 2.54 (New Jersey). Values larger than 2 are obtained for stations from mid-latitude, humid environments. Values close to 1 are obtained for the two maritime, high latitude stations. An intermediate value of 1.68 is obtained for the Marshall Islands site.

Mean drop arrival rate ranges from near 400 drops $m^{-2}s^{-1}$ in Oregon and Alaska to 1791 drops $m^{-2}s^{-1}$ in the Marshall Islands. The average mean diameter ranges from 1.07 mm in Alaska to 1.34 mm in Oregon. The average value of the process $1 + c(t)^2$ is remarkably similar over the five sites, the largest value being 1.07 and the smallest 1.06. The term $1+c(t)^2$ and the mean diameter provide the information that is needed to compute the average volume of a raindrop (see equation (5)). Qualitatively, the term $1 + c(t)^2$ incorporates the effects of variability of drop diameters on rainfall rate. If there is no variability in drop diameter, i.e. if all drops are of the same size, the coefficient of variation process c(t) equals 0 and $1 + c(t)^2$ equals 1.

The contrast in mean rain rate between North Carolina (6.33 mm h^{-1}) and New Jersey (3.66 mm h^{-1}) can be attributed to the difference in mean drop arrival rate

(1081 drops $m^{-2}s^{-1}$ in North Carolina versus 717 drops $m^{-2}s^{-1}$ in New Jersey) and not to differences in drop diameter characteristics (the mean diameter for North Carolina is 1.19 mm versus 1.18 mm for New Jersey). The same holds for differences in rain rate between the mid-latitude stations and the tropical station. The Marshall Islands site has a somewhat larger mean diameter than North Carolina (1.27 mm to 1.19 mm) but a much larger drop arrival rate (1791 drops $m^{-2}s^{-1}$). The differences in mean rain rate between Alaska (1.28 mm h^{-1}) and Oregon (2.28 mm h^{-1}) arise from differences in mean diameter rather than differences in drop arrival rate. These stations have very similar drop arrival rates (407 and 410 drops $m^{-2}s^{-1}$), but Oregon has a large mean diameter (1.34 mm - as noted above, the largest of the five sites) and Alaska has a small mean diameter (1.07 mm - the smallest of the five sites).

The product representation of rainfall rate in terms of raindrop processes (5) makes it natural to examine the relationship between rainfall rate and raindrop processes through their logarithms. We have

$$\tilde{R}(t) = \tilde{\lambda}(t) + 3\tilde{m}(t) + 3\tilde{c}(t) + C \tag{6}$$

where, $\tilde{R}(t) = ln(R(t))$, $\tilde{\lambda}(t) = ln(\lambda(t))$, $\tilde{m}(t) = ln(m(t))$, $\tilde{c}(t) = ln(1 + c(t)^2)$ and C is the constant $ln[6\pi 10^{-4}]$. The variance of log rainfall rate can be decomposed into terms involving component raindrop processes:

$$\begin{aligned} Var(\tilde{R}(t)) &= Var(\tilde{\lambda}(t)) + 9Var(\tilde{m}(t)) + 9Var(\tilde{c}(t)) \\ &\quad +6Cov(\tilde{\lambda}(t), \tilde{m}(t)) + 6Cov(\tilde{\lambda}(t), \tilde{c}(t)) + 18Cov(\tilde{m}(t), \tilde{c}(t)) \end{aligned} \tag{7}$$

Variability of log rainfall rate can be examined in terms of sums of variance components associated with raindrop processes (using (7)). Table 2 provides an analysis of variance of rainfall rate in terms of raindrop processes. Results are presented for the same sites as in Table 1 (values represent variance/covariance terms for quantities given in the left column). For North Carolina and New Jersey, the variance component associated with drop arrival rate is 3 times larger than the variance component associated with mean diameter (compare, for example, the temporal pattern of rainfall rate and raindrop processes for the North Carolina storms in figures 1 and 2). For the Marshall Islands site the ratio increases to 5:1. For the Alaska and Oregon sites, however, the two terms are close in magnitude (the mean diameter term is approximately 75 percent of the drop arrival rate term). Variability of diameter CV is small for all sites, accounting for less than 3 percent of the variability of log rainfall rate.

Covariance terms involving drop arrival rate and mean diameter account for 10 percent or more of the total variance of log rainfall rate. Notably, the sign of the covariance terms differs among the five sites. For Alaska and Oregon these covariance terms are negative. For the other sites these covariance terms are positive.

The final two lines of Table 2 compare the variance of log rainfall rate computed by summing the variance components associated with raindrop process (denoted "Total") and the variance of log rainfall rate computed directly from rainfall rate data. The close agreement between model variance computations and the sample variances provides further evidence that the raindrop process model can accurately represent distributional properties of rainfall rate.

The representation of rainfall rate in (6) makes the lognormal distribution an attractive choice for modeling rainfall rate. Equation (6) implies that rainfall rate has a lognormal distribution whenever $\tilde{\lambda}(t)$, $\tilde{m}(t)$, and $\tilde{c}(t)$ have a multivariate normal distribution (see Anderson [1985]). Consequently, the lognormal assumption for rainfall rate can be readily examined in terms of distributional properties of raindrop processes. The lognormal distribution will be used extensively in statistical analyses presented in the following sections.

Model development presented in this section pertains to a single storm. Statistical analyses will be based on samples of rainfall rate and raindrop processes from multiple storms. The following notation represents the sample of rainfall rate and raindrop processes that will be used in the following sections:

n = number of storms.

$R_i(t)$ = rainfall rate (mm h^{-1}) for time t of ith storm.

$\lambda_i(t)$ = drop arrival rate (drops $m^{-2}s^{-1}$) for time t of ith storm.

$m_i(t)$ = mean diameter (mm) for time t of ith storm.

$c_i(t)$ = coefficient of variation of drop diameter for time t of ith storm.

3 Lognormal Model for Rainfall Rate

In this section statistical models are described for representing rainfall rate $\{R_i(t)\}$ during the course of a storm. Two principal issues are examined in this section:

- What is the marginal distribution of rainfall rate?
- What is the temporal correlation structure of rainfall rate?

Examination of the marginal distribution of rainfall rate focuses on two classes of lognormal models and their relationship to distributional properties of raindrop processes. The analyses of correlation structure of rainfall rate focus on the link to distributional assumptions on rainfall rate and the relationship with covariance properties of raindrop processes. Analyses are carried out in this section for the North Carolina site, due to the length and quality of record. Results for North Carolina are qualitatively similar to those for other sites.

3.1 Lognormal Model - Constant Parameters

A simple lognormal model for rainfall rate, in which model parameters do not vary from storm to storm or during the course of a storm, is examined below. The notation:

$$\tilde{R}_i(t) \sim N(\mu, \sigma^2) \tag{8}$$

indicates that log rainfall rate has a normal distribution with mean μ and variance σ^2 and that these parameters do not vary with the storm index i or with the time index t.

Figure 4 shows lognormal quantile - quantile (Q-Q) plots for rainfall rate and raindrop processes for the North Carolina data (the notation CV1 is used for the process $ln(1 + c(t)^2)$). The lognormal assumption is quite reasonable for the central portion of the rainfall rate distribution. Clear departures appear, however, in the lower tail of the distribution.

The lognormal Q-Q plot for drop arrival rate is qualitatively similar to the lognormal Q-Q plot for rainfall rate. This is not the case for variability in drop diameter. The term representing variability in drop diameter $ln(1 + c(t)^2)$ exhibits marked departures from the normal distribution. To the extent that variability in drop diameter affects rainfall rate, it will lead to departures from the lognormal assumption; recall, however, that in table 2 it was shown that variability in drop diameter has a relatively small impact on variability in rainfall rate.

The autocorrelation function of log rainfall rate is denoted as follows:

$$\rho(t) = \frac{E[(\tilde{R}_i(s+t) - E[\tilde{R}_i(s+t)])(\tilde{R}_i(s) - E[\tilde{R}_i(s)])]}{E[(\tilde{R}_i(s) - E[\tilde{R}_i(s)])^2]} \tag{9}$$

Figure 5 shows the sample autocorrelation function of log rainfall rate and raindrop processes for North Carolina. The autocorrelation at lag t is obtained by comparing observations separated by t minutes from all storms and using sample mean and variance computed from the entire data set. The autocorrelation of log rain rate drops from 0.84 at a time lag of 1 minute to 0.35 at 20 minutes. The autocorrelation fluctuates in a narrow band about 0.35 for lags ranging from 20 minutes to 60 minutes. Raindrop processes exhibit similar features. The only marked difference is the smaller autocorrelation values for the process $\tilde{c}(t)$ representing drop diameter variability.

The autocorrelation functions exhibit two principal features, a rapid decline for short time lags and a period of near constant positive correlation. The latter feature has been interpreted as evidence for long-term persistence. The autocorrelation analysis will be reexamined below.

Interaction of raindrop processes could potentially affect the distribution of rainfall rate, especially autocorrelation properties of rainfall rate. Figure 6 shows lagged cross-correlation plots for raindrop processes. Cross-correlation values are relatively small (near 0.20) but nearly constant with varying time lags. These results will also be revisited below.

3.2 Lognormal Model - Storm Varying Parameters

The preceding model does not allow for systematic variability of rainfall characteristics from storm to storm. A simple way of generalizing the preceding model is to allow the mean and variance to vary from storm to storm:

$$\tilde{R}_i(t) \sim N(\mu_i, {\sigma_i}^2) \tag{10}$$

The parameter μ_i represents the mean of log rainfall rate for the ith storm; the parameter ${\sigma_i}^2$ represents the variance of log rainfall rate for the ith storm. Mean rainfall rate for a storm is given by:

$$E[R_i(t)] = e^{\mu_i + 0.5{\sigma_i}^2} \tag{11}$$

The coefficient of variation of rainfall rate is given by:

$$CV(R_i(t)) = [e^{{\sigma_i}^2} - 1]^{1/2} \tag{12}$$

Figure 7 illustrates: 1) storm to storm variability in mean rain rate and rain rate CV and 2) dependence of rain rate CV on mean storm rain rate. In the figure the sample mean of rain rate for a storm is plotted versus the sample coefficient of variation of rain rate for the storm for the North Carolina site (88 storms), Marshall Island site (42 storms), and the Alaska site (48 storms). Storms with small mean rainfall rates tend to have relatively low variablity. The largest variability is seen in storms with moderate mean rainfall rate. It is plausible that rain rate CV depends in a systematic fashion on mean rain rate. Moderate mean rain rates can result from alternating periods of high and low rain rate (which yields a high CV for rain rate) or from periods of steady, moderate rain (which yields a low CV for rain rate). By contrast, storms with low mean rain rates (and possibly storms with high mean rain rates) may be constrained in the range of rain rate values that occur during the storm.

Figure 8 shows lognormal Q-Q plots for centered and scaled rainfall rate and raindrop processes; log rainfall rate values are centered by the mean of the log rainfall rate values for the storm and scaled by the standard deviation of log rainfall rate for the storm. The lognormal assumption is quite good for this case, even in the tails of the distributions.

Storm to storm variation in the mean and/or variance of log rainfall rate can have important implications for autocorrelation analyses. If the model of (10) holds and we use the sample mean and variance over the entire period of record, errors are introduced into the sample autocorrelation function. The following computation illustrates the nature and magnitude of these errors. Let μ be an arbitrary constant. Then under the assumption that (10) holds,

$$\frac{E[(\tilde{R}_i(s+t)-\mu)(\tilde{R}_i(s)-\mu)]}{E[(\tilde{R}_i(s)-\mu)^2]} = \frac{\rho(t)}{1+\frac{(\mu-\mu_i)^2}{{\sigma_i}^2}} + \frac{(\mu-\mu_i)^2}{{\sigma_i}^2+(\mu-\mu_i)^2} \qquad (13)$$

This result illustrates that if improper sample mean and variance are used in the computation of a sample autocorrelation, the result is to scale the true time dependence of the autocorrelation function by a constant larger than 1 (the first term of (13)) and to add a constant $shelf$ to the sample autocorrelation (the second term of (13)).

Figure 9 shows the sample autocorrelation function computed for centered and standardized log rainfall rate and raindrop processes. The autocorrelation function decays rapidly to 0 (within approximately 10 minutes). The shelf at 0.35 has disappeared. In figure 10, the lagged cross-correlation properties of raindrop processes (see figure 6) are reexamined using storm-varying parameters. The near-constant positive cross-correlation has been replaced by a near-constant 0 cross-correlation. From this analysis, there is little evidence for joint dependence among raindrop processes.

The results for the lognormal model with storm-dependent mean and variance have appealing features. The lognormal fit is clearly better in this case. Furthermore, it is not necessary to resort to long-term dependence arguments to explain autocorrelation and lagged cross-correlation results.

The preceding models do not allow for systematic variation of rainfall rate over the course of the storm. Figure 11 shows a scatterplot of rainfall rate versus time from onset of the storm. There is not a clear dependence on time position within a storm. Previous studies, however, have provided evidence suggesting that storm rainfall exhibits systematic structure over its life history (see, for example, Waldvogel [1974] and Woolhiser and Osborn [1985]).

4 Lognormal Cascade Model of Rainfall

In this section, the dependence of rainfall rate R(t) on the time integration period Δt is examined. It is shown that the lognormal models presented in the previous section can be naturally extended to explicitly account for averaging time interval. To emphasize the dependence of rainfall rate on the averaging time interval the notation for rainfall rate will be expanded as follows:

$$R(t, \Delta t) = \frac{A(t) - A(t - \Delta t)}{\Delta t} \tag{14}$$

Rainfall rate, as defined in (14), can be represented in terms of raindrop processes as follows:

$$R(t, \Delta t) = [6\pi 10^{-4}]\Lambda(t, \Delta t)M(t, \Delta t) \tag{15}$$

where

$$\Lambda(t, \Delta t) = \Delta t^{-1} \int_{t-\Delta t}^{t} \lambda(u)du \tag{16}$$

and

$$M(t, \Delta t) = \frac{\int_{t-\Delta t}^{t} \lambda(u)m(u)^3(1 + c(u)^2)^3 du}{\int_{t-\Delta t}^{t} \lambda(u)du} \tag{17}$$

The quantity $\Lambda(t, \Delta t)$ is the average drop arrival rate during [t-Δt,t]; the quantity $M(t, \Delta t)$ is the average volume of a drop during [t-Δt,t].

The 1 minute raindrop data will serve as the basis for empirical analyses in this section. The actual time scale of these observations is somewhat different from 1 minute due to the sampling characteristics of the instrument. An implicit assumption of the empirical analyses is that the partial minute observations from the raindrop camera represent conditions during the entire minute.

The constant parameter lognormal model of section 3 can be extended as follows:

$$\tilde{R}(t, \Delta t) \sim N(\mu + \eta ln(\Delta t), \sigma^2 - \gamma ln(\Delta t)) \tag{18}$$

where $\tilde{R}(t, \Delta t) = ln(R(t, \Delta t))$ and $\gamma \geq 0$. In this model log rainfall rate has a normal distribution with mean $\mu + \eta ln(\Delta t)$ and variance $\sigma^2 - \gamma ln(\Delta t)$. The parameterization of (18) allows both the mean and coefficient of variation of rainfall rate to depend on Δt.

Similar models have been used for scale analysis of hydrometeorological processes and termed lognormal cascade models or lognormal multiscaling models (see Gupta and Waymire [1990 and 1992], Lovejoy and Schertzer [1990], and Smith [1992]). If the parameter γ equals 0, the model simplifies to a lognormal simple scaling model. The terms simple scaling and multiscaling are used in their *wide sense* formulation (see Gupta and Waymire [1990]).

Mean rainfall rate for the model is given by:

$$E[R(t, \Delta t)] = e^{\mu + 0.5\sigma^2} \Delta t^{\eta - 0.5\gamma} \tag{19}$$

or equivalently:

$$ln(E[R(t, \Delta t)]) = \mu + 0.5\sigma^2 + (\eta - 0.5\gamma) ln(\Delta t) \tag{20}$$

The coefficient of variation of rainfall rate is given by:

$$CV(R(t, \Delta t)) = [e^{\sigma^2} \Delta t^{-\gamma} - 1]^{1/2} \tag{21}$$

or equivalently:

$$ln(1 + CV(R(t, \Delta t))^2) = \sigma^2 - \gamma ln(\Delta t) \tag{22}$$

A diagnostic feature of the lognormal simple scaling model is that the coefficient of variation of rainfall rate does not depend on the averaging time interval Δt.

Equation (19) implies that mean rainfall rate varies with the averaging time interval Δt if γ does not equal 2η. Equations (3) and (14) can be used directly to conclude that if raindrop processes are stationary in time within a storm and mutually independent, then

$$E[R(t,\Delta t)] = [\frac{\pi}{6}10^{-6}]\lambda_{av}m_{av}c_{av} \tag{23}$$

where λ_{av} is the stationary mean of the drop arrival rate process $\lambda(u)$, m_{av} is the stationary mean of the drop diameter process m(u) and c_{av} is the stationary mean of the process $1 + c(u)^2$. Under the assumption that the raindrop processes are stationary, the mean rainfall rate will not depend on the averaging time interval and we can conclude that

$$\gamma = 2\eta. \tag{24}$$

Equation (22) implies that $ln(1 + CV(R(t,\Delta t))^2)$ (which will be denoted CV2 of rainfall rate) is a linear function of $ln(\Delta t)$. For the North Carolina rainfall rate data, sample values of CV2 decrease monotonically with averaging time interval from 1 to 30 minutes (figure 12). Clearly, the lognormal simple scaling model is not valid for rainfall rate at accumulation time intervals ranging from 1 - 30 minutes. The pattern of decline of CV2 for rainfall rate $R(t,\Delta t)$ is quite similar to that of drop arrival rate $\Lambda(t,\Delta t)$ (figure 12). Both exhibit a slightly convex dependence on $ln(\Delta t)$. By contrast, CV2 for diameter volume $M(t,\Delta t)$ appears to more closely exhibit linear dependence on $ln(\Delta t)$, suggesting closer agreement with the lognormal multiscaling model.

The results suggest the following conclusions:

- The lognormal simple scaling model is not appropriate for modeling rainfall rate at time scales ranging from 1 to 30 minutes.
- Modest departures from the lognormal cascade model are suggested for North Carolina rainfall rate data, possibly involving higher order dependence on $ln(\Delta t)$.

- Climates with drop diameter dominance (as represented by Alaska and Oregon) may be better represented by a logormal cascade model than climates with drop arrival rate dominance.

The lognormal cascade model can be generalized to represent the case in which mean and variance of rainfall rate vary from storm to storm:

$$\tilde{R}_i(t, \Delta t) \sim N(\mu_i + \eta ln(\Delta t), {\sigma_i}^2 - \gamma ln(\Delta t)) \tag{25}$$

Moments for this model differ from (19) and (21) as follows:

$$E[R_i(t, \Delta t)] = e^{\mu_i + 0.5{\sigma_i}^2} \Delta t^{\eta - 0.5\gamma} \tag{26}$$

$$CV(R(t, \Delta t)) = [e^{{\sigma_i}^2} \Delta t^{-\gamma} - 1]^{1/2} \tag{27}$$

Statistical inference for the model can be carried out by noting that

$$\tilde{R}_i(t, \Delta t) - E[ln(R_i(t, \Delta 1)] \sim N(\eta ln(\Delta t), {\sigma_i}^2 - \gamma ln(\Delta t)) \tag{28}$$

5 Summary and Conclusions

Temporal variation of rainfall has been examined for *short* time scales, i.e. time scales less than or equal to 1 minute, using a raindrop process representation of rainfall and drop-size data. In Section 2 it is shown that rainfall rate can be represented quite accurately in terms of temporal variation of drop arrival rate, mean diameter and drop diameter CV. Furthermore, it is shown that climatological variability of rainfall rate can be related to the relative dominance of drop arrival rate properties and drop diameter properties.

In section 3, two models of rainfall rate are analyzed. In the first, rainfall rate has a lognormal distribution with parameters that are fixed, both for the duration of a storm and from storm to storm. In the second model, the lognormal parameters are fixed for the duration of a storm but vary from storm to storm. For the first model, empirical analyses exhibit departures from the lognormal assumption in the tails of the distribution. Furthermore, sample autocorrelation functions suggest long-term dependence. For the lognormal model with parameters that vary from storm to storm, the lognormal fit is quite good, even in the tails of the distribution. For this model, the positive shelf in autocorrelation and lagged cross-correlation analyses disappears.

Temporal scaling properties of rainfall rate and raindrop processes are examined in section 4 by extending the lognormal models of section 3 to lognormal cascade models that explicitly accommodate time-averaging in rainfall rate. Empirical analyses strongly support the lognormal cascade model over a lognormal simple scaling model. Results suggest modest departures from the lognormal cascade model for the North Carolina rainfall rate data, possibly involving higher order dependence on averaging time interval.

Results of this paper can be used to make concrete recommendations concerning models for short-term rainfall rate. For practical applications, rainfall rate during a storm can be usefully represented by a lognormal Markov process. It is preferable to allow the mean and variance of rainfall rate to vary in time. A simple and accurate representation is to allow the lognormal parameters to vary from storm to storm. In this case, the autocorrelation of rainfall rate decays rapidly, with values near 0.8 for a time lag of 1 minute and values near 0 for time lags greater than 10 minutes. Model parameters depend on the averaging time interval; a model designed for 5 minute accumulations will have different parameters from a model designed for 1 minute or 30 minute accumulations. Dependence on the modeling time interval can be represented using the lognormal cascade formulation.

Models of short-term rainfall rate can be used independently or embedded as a component of a larger model, for example in representing short-term variablity of rainfall within a "pulse" of a Neyman Scott rectangular pulse model (Rodriguez-Iturbe [1986]). The developments of this paper also show that lognormal models for raindrop processes underlie the lognormal models for rainfall rate. Starting with lognormal models of raindrop processes, coupled models of rainfall rate and related processes, such as radar reflectivity factor and rainfall kinetic energy flux, can be developed. Thus the raindrop process representation provides a common modeling framework for integrating remote sensing observations with land surface processes.

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APPENDIX

The formulas for rainfall accumulation and rainfall rate (equations 3 - 5) are based on large sample approximations. We sketch below the arguments that are used in establishing these results.

The case in which drop arrival rate, $\lambda(u)$, varies in time but mean diameter, m(u), and diameter CV, c(u), do not will be examined first. It follows from the strong law of large numbers that

$$\eta(t)^{-1} \sum_{i=1}^{\eta(t)} {D_i}^3 \approx E[D^3] \tag{29}$$

for large values of $\eta(t)$. In this case we also have

$$\eta(t) \approx \int_0^t \lambda(u)du \tag{30}$$

This leads to the approximation

$$\sum_{i=1}^{\eta(t)} {D_i}^3 \approx [m^3(1+c^2)^3] \int_0^t \lambda(u)du \tag{31}$$

which is the desired result in the case c(u) = c and m(u) = m for all times u. Drop arrival rate values range from 100 - 10,000 drops $m^{-2}s^{-1}$. For time intervals and areas of practical interest the large sample assumptions should be quite good. The preceding result uses lognormality of drop diameters.

If mean diameter and diameter CV vary with time, the following arguments are needed. In this case the strong law of large numbers again implies that

$$\eta(t)^{-1} \sum_{i=1}^{\eta(t)} {D_i}^3 \approx E[D^3] \tag{32}$$

but in this case drop diameter has a "mixed" distribution. The mean drop volume is obtained by averaging the time-varying mean drop volumes $m(u)^3(1+c(u)^2)^3$. Time-averaging of the mean drop volumes must take account of the time-varying rate of occurrence of raindrops. The mean drop volume thus becomes

$$E[D^3] = \frac{E[\sum_{i=1}^{\eta(t)} {D_i}^3 | \lambda(u), m(u), c(u), u \geq 0]}{E[\eta(t) | \lambda(u), m(u), c(u), u \geq 0]}. \tag{33}$$

An alternative represntation of $\sum_{i=1}^{\eta(t)} {D_i}^3$ is the following:

$$\sum_{i=1}^{\eta(t)} {D_i}^3 = \int_0^t \int_0^\infty x^3 dM(x,u) \tag{34}$$

where M is a Poisson process on $[0,\infty) \times [0,\infty)$ with intensity function

$$\Lambda(t,x) = \lambda(t) f_t(x) \tag{35}$$

and $f_t(x)$ is the lognormal density function of drop diameter at time t.

It follows from Karr [1986] that

$$\begin{aligned}
E[\sum_{i=1}^{\eta(t)} D_i{}^3|\lambda(u), m(u), c(u), u \geq 0] &= E[\int_0^t \int_0^\infty x^3 dM(x,u)|\lambda(u), m(u), c(u), u \geq 0] \\
&= \int_0^t \int_0^\infty x^3 \Lambda(u,x) dx du \\
&= \int_0^t \int_0^\infty x^3 \lambda(u) f_t(x) dx du \\
&= \int_0^t \lambda(u) [\int_0^\infty x^3 f_t(x) dx] du \\
&= \int_0^t \lambda(u) E[D_i{}^3 \mid T_i = u] du \\
&= \int_0^t \lambda(u)[m(u)^3(1 + c(u)^2)^3] du \qquad (36)
\end{aligned}$$

In the preceding derivation T_i is the arrival time of the ith raindrop.

Because $\eta(t)$ is a Cox process directed by $\lambda(t)$ (see Karr [1986]) it follows that

$$E[\eta(t)|\lambda(u), m(u), c(u), u \geq 0] = \int_0^t \lambda(u) du \qquad (37)$$

Combining the previous computations we have:

$$\eta(t)^{-1} \sum_{i=1}^{\eta(t)} D_i{}^3 \approx \frac{\int_0^t \lambda(u)[m(u)^3(1 + c(u)^2)^3] du}{\int_0^t \lambda(u) du} \qquad (38)$$

Using (30) once more we have:

$$\sum_{i=1}^{\eta(t)} D_i{}^3 \approx \int_0^t \lambda(u)[m(u)^3(1 + c(u)^2)^3] du \qquad (39)$$

which is the desired result in the case that drop arrival rate, mean diameter and diameter CV are each time varying stochastic processes.

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List of Figures

	NC	NJ	ALA	ORE	M.I.
$E[R(t)]$	6.33	3.66	1.28	2.28	11.06
$CV(R(t))$	2.31	2.54	1.26	1.06	1.68
$E[\lambda(t)]$	1081	717	407	410	1791
$CV(\lambda(t))$	1.54	1.69	0.92	0.95	1.19
$E[m(t)]$	1.19	1.18	1.07	1.34	1.27
$CV(m(t))$	0.24	0.20	0.23	0.22	0.19
$E[1+c(t)^2]$	1.07	1.06	1.07	1.07	1.07
$CV(1+c(t)^2)$	0.05	0.05	0.05	0.05	0.04

Table 1: Summary statistics for rainfall rate and raindrop processes for sites in North Carolina, New Jersey, Alaska, Oregon, and the Marshall Islands. Units are as follows: rainfall rate - mm h^{-1}, drop arrival rate - drops $m^{-2}s^{-1}$, mean diameter - mm, $1 + c(t)^2$ - dimensionless.

	NC	NJ	ALA	ORE	M.I.
$ln(\lambda(t))$	1.43	1.07	0.58	0.54	1.53
$9ln(m(t))$	0.55	0.34	0.44	0.42	0.33
$9ln(1+c(t)^2)$	0.02	0.02	0.02	0.02	0.01
$6ln(\lambda(t))ln(m(t))$	0.46	0.33	-0.09	-0.32	0.20
$6ln(\lambda(t))ln(1+c(t)^2)$	0.05	0.03	0.01	0.02	0.06
$18ln(m(t))ln(1+c(t)^2)$	0.04	0.02	0.01	-0.03	0.04
Total	2.54	1.81	0.97	0.65	2.17
$Var(ln(R(t))$	2.55	1.82	0.96	0.65	2.16

Table 2: Variance components of log rainfall rate for five sites from Table 1.

Figure 1: State estimates of raindrop processes for a storm in North Carolina on a) April 9, 1961 and b) May 9, 1961.

Figure 2: Model estimates of rainfall rate for 2 North Carolina Storms (April 9, 1961 and May 9, 1961). Dashed lines represent model estimates; solid lines represent observed rainfall rate.

Figure 3: Dependence of model error on rainfall rate and raindrop processes for 4,741 1 minute samples from North Carolina.

Figure 4: Lognormal Q-Q plots for rainfall rate and raindrop processes in North Carolina.

Figure 5: Sample autocorrelation function of rainfall rate and raindrop processes for North Carolina.

Figure 6: Sample lagged cross-correlation functions for raindrop processes for North Carolina. Solid line represents CV1; dashed line represents mean diameter; dotted line represents drop arrival rate.

Figure 7: Mean rainfall rate versus CV of rainfall rate for storms in North Carolina, Marshall Islands and Alaska.

Figure 8: Lognormal Q-Q plots for standardized rainfall rate and raindrop processes in North Carolina.

Figure 9: Sample autocorrelation function of standardized rainfall rate and raindrop processes for North Carolina.

Figure 10: Sample lagged cross-correlation functions for standardized raindrop processes for North Carolina. Solid line represents CV1; dashed line represents mean diameter; dotted line represents drop arrival rate.

Figure 11: Scatterplot of rainfall rate versus time (in minutes) from storm onset for North Carolina storms. Solid line is a LOWESS (locally weighted scatterplot smoothing, see Becker et al. [1988]) representation of the relationship beteween rainfall rate and time.

Figure 12: Scatterplot of sample CV2 versus accumulation time interval for North Carolina storms.