APPROVAL VOTING WITH ENDOGENOUS CANDIDATES

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Abstract. We present a formal model of political competition under approval voting which allows for endogenous candidate entry. Our analysis yields a number of novel insights. First, we develop a notion of sincere voting behavior under approval voting, called *relative sincerity*. We then show that the relatively sincere voting behavior is consistent with the strategic calculus of voting. Second, we show that in a one dimensional model with distance preferences, equilibria in relatively sincere strategies, with serious candidates always generate outcomes close to the median voter. Moreover, approval voting satisfies Duverger's Law in the sense that there are at most two winning positions! Finally, we extend our analysis to arbitrary policy spaces. In the general setting, approval voting is shown to be susceptible to the same kinds of problems as the plurality rule such as the possibility of extreme outcomes, failure to elect the Condorcet winner and existence of spoiler candidates.

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1. Introduction

The objective of this paper is to study a model of endogenous candidacy under Approval Voting (hereafter AV) and to compare its outcomes to those under the plurality rule.

AV is a non-rank, scoring method of voting. Under this method a voter can vote for as many alternatives as he wishes with the restriction that each alternative can receive at most one vote. All votes count equally and the alternative receiving the highest number of votes is chosen to be the winner. AV is used by several academic and professional bodies to elect their office-bearers.[1] Many scholars of electoral systems have recommended that AV be used for political elections as well and regard it to be '*the* electoral reform of the 21st century' (see Brams, 1980; 105). There is a large body of literature that studies the electoral outcomes under AV and compares them vis-a-vis the plurality rule.[2] However, the problem of endogenous candidate entry has so far been ignored in the literature.[3]

[1]For example, the Fellows of the Econometric Society and the President of the Social Choice and Welfare Society (SCWS) are elected under AV (see Brams and Fishburn (2000) for a description of the first election of a President of the SCWS under AV that took place in 1999); the Institute of Electrical and Electronics Engineers, the American Statistical Association or the Mathematical Association of America are some of the professional organizations who have adopted this method (see Brams and Fishburn (1988) and Brams and Nagel (1991) for a discussion of some of those elections); AV is also used in electing the Secretary General of the United Nations, in elections in some Eastern European countries and in voting over multiple-related initiatives in some US States.

[2]We review the main findings of this literature in Section 2.

[3]Many scholars have recognized the importance of endogenizing the set of candidates, as Fishburn and Brams (1981c; 426) note: "Several people have also expressed concern about how Approval voting would affect who enters an election and how it would influence candidates' strategies. Although we do not address this concern, it surely deserves examination."

The present work seeks to fill this gap by incorporating the politicians' entry decisions into the electoral game. For this purpose, we use the 'citizen-candidate' approach to electoral competition.[4] Under the citizen-candidate approach, the political process is modeled as a three stage game. At the first stage, policy-motivated candidates decide whether or not to run for office at a cost. At the second stage, the voters vote over the set of candidates. At the third stage, the winner implements his preferred policy. Both candidate entry and voting are modeled as strategic decisions made by rational agents.

There are two advantages to using the citizen-candidate approach for the comparative analysis of electoral systems. First, the citizen-candidate approach allows us to understand the effect of a voting rule on the incentives it creates for candidates to enter the political race. This is important given that every non-dictatorial voting procedure that satisfies unanimity is open to strategic entry or exit by candidates (see Dutta, Jackson, and Le Breton (2001)). Second, while the Downsian analysis of AV has been limited to the unidimensional model with three parties, the citizen-candidate approach enables us to go beyond this restrictive setting and makes it possible for us to handle the multi-dimensional policy space with an arbitrary number of potential candidates.

Our analysis yields a number of interesting results. The first set of results concerns the notion that AV encourages sincere voting. Under AV, the concept of sincerity is taken to mean that a citizen who votes for a candidate k must also vote for any other candidate j whom he prefers to k. However, there are many strategies which can be regarded as sincere under this definition. We propose a further refinement of the voting behavior, viz.- *relatively sincere voting behavior*. An agent is said to vote relatively sincerely if, given others' voting strategies, he votes (does not vote,

[4]The citizen-candidate approach was pioneered by Osborne and Slivinski (1996) and Besley and Coate (1997). Our model is closer in spirit to the latter paper since we model candidate entry as well as voting as strategic decisions.

resp.) for all those candidates who give him a strictly higher (lower, resp.) payoff than the expected payoff from the outcome of the election. We show that a voter's best-response set always contains a *relatively sincere voting strategy*. We also prove, in the one-dimensional model with single-peaked preferences, the existence of a voting equilibrium in relatively sincere strategies.[5]

The second set of results concerns the electoral process under AV when the policy space is unidimensional and the voters have distance preferences. Our analysis shows that, while there may be a multiplicity of candidates under AV, there are at most two winning positions. We also show that the outcomes of equilibria in which, (*i*) there are no spoiler candidates[6] and (*ii*) which satisfy relative sincerity, are arbitrarily close to the median voter's ideal position as the entry costs become small. In contrast to this, under the plurality rule there always exist equilibria which are extreme. However, this result suggesting policy moderation under AV does not extend to the equilibria involving spoiler candidates. Hence, our analysis suggests that the role of AV in generating policy moderation depends upon the plausibility of equilibria with spoiler candidates.

Our third set of results characterizes the political equilibria under AV in a general setting. We find that the outcomes under AV and those under the plurality rule are generally distinct, *even* in elections with only two candidates! In particular, neither is a subset of the other. This may come as a surprise, especially since AV and the plurality rule are equivalent methods when there are two *exogenously given* alternatives. However, when one considers the fact that candidate entry is endogenous, the equivalence no longer holds. This highlights a methodological problem in comparing electoral systems by studying their performance over an exogenous set of alternatives. We also find that AV is vulnerable to the same kinds of problems as the plurality rule

[5]Our refinement is intuitively plausible and finds some support in the empirical studies of voting behavior under AV (e.g., see Merrill and Nagel (1987)).

[6]i.e. candidates who run in the race not to win but to prevent others from winning

such as non-majoritarian policy outcomes, failure to elect the Condorcet winner and presence of spoiler candidates.

The organization of the remainder of the paper is as follows. In section 2 we review the current literature. In section 3 we present the model and develop various refinements of voting behavior. In section 4 we study our model in the context of a unidimensional policy space with distance preferences. Section 5 extends our analysis to arbitrary policy spaces. Section 6 concludes. All proofs are in the appendix.

2. Related Literature

The present work contributes to a large literature on the comparative properties of various electoral systems. There are two strands of this literature: (*i*) the literature on the properties of voting behavior over a fixed set of alternatives, and (*ii*) Downsian models of political competition under alternative voting rules.

The first strand of the literature was pioneered by Brams and Fishburn (1978). This paper compares different single-ballot, non-rank voting rules under dichotomous and trichotomous preferences.[7] The authors show that when the preferences are dichotomous, each voter has a unique undominated strategy under AV and it always elects the Condorcet winner. This is not true of the other voting rules. Moreover, when the preferences are trichotomous, AV is the only system under which the set of sincere voting strategies is equivalent to the set of undominated strategies. Brams and Fishburn (1981a) shows that AV dominates the plurality rule in the sense that if the latter system elects the Condorcet winner, then the former must elect it as well, but the converse is not true. They also show that whenever a Condorcet winner

[7] Both AV and the plurality rule are special cases of single-ballot, non-rank voting rules. A voter is said to have dichotomous preferences if the set of alternatives can be partitioned into two subsets, say, M and L, such that all elements in M are strictly preferred to those in L and the voter is indifferent between all alternatives within the sets M and L. Under trichotomous (multichotomous, resp.) preferences, there are three (more than three, resp.) such partitions.

exists, there exists a sincere strategy profile under AV which elects it. Brams and Fishburn (1981b) and (1981c) show that in any optimal voting strategy, an expected utility maximizer should vote for all the serious contenders that give him a utility higher than the expected utility. Hence, voting must be sincere on the set of serious contenders.[8]

Scholars have criticized the above approach on several grounds. Some rightly argue that the preference structure that admits only dichotomous and trichotomous preferences is restrictive. Others have pointed out that even if the voters voted sincerely, they still need to make strategic calculations in deciding how many candidates to vote for.[9] Cox (1984) which looks at three candidate elections in two-member districts[10] in England between 1832 and 1867 finds evidence supporting strategic behavior by voters. The importance of strategic voting is further made clear in De Sinopoli (1999) where the author shows that even though there exists a sincere strategy that implements the Condorcet winner, such a strategy need not be consistent with sophisticated voting in the sense that it fails to survive iterated elimination of weakly dominated strategies. We draw on this critique and focus on equilibrium voting strategies.

The second strand of literature uses the Downsian framework to study various electoral systems. Cox (1985) considers a unidimensional, Downsian model with three political parties. He shows that there exists a unique Nash equilibrium in which all the parties adopt the median voter's ideal position. However, the result does not generalize to more than three parties. Cox (1987) compares the outcomes under the plurality rule to those obtained under AV. He shows that while with two parties, both systems generate the same, centrist equilibria, when there are three

[8]There seems to be here some confusion since several authors have subsequently inferred that voting must be sincere over the set of *all* candidates rather than the set of serious contenders only.

[9]See for example Niemi (1984), Arrington and Brenner (1984) or Saari and Van Newenhizen (1988). Also see Tullock(1979).

[10]Each voter could cast up to two votes without accumulating them and therefore the electoral rule was equivalent to approval voting over three candidates.

parties, the equilibria under AV are centrist while those under the plurality rule are not. Our analysis shows that even in two candidate equilibria, the set of equilibrium outcomes differ under the two systems and AV produces more moderate outcome as compared to the plurality rule. Myerson and Weber (1993) also compare different electoral systems with three Downsian parties. They look at equilibria under which voters hold rational expectations about their being pivotal over every possible pair of candidates. They find that, under AV, all the serious contenders are located at the position of the median voter. In contrast, the plurality rule imposes little restriction on the position of the winning candidates, making it possible for an extremist to be elected. Our analysis generalizes this result in the sense that we consider an arbitrary number of potential entrants. However, our analysis also provides a important caveat, viz. the equilibria must not have spoiler candidates.[11]

3. The Model

Consider a polity $\mathcal{N}$ consisting of a finite number of citizens, indexed by $\ell = 1, ..., |\mathcal{N}|$. This polity must elect a public official who will be in charge of implementing a policy. Let $\mathcal{A}$ denote the finite and non-empty set of feasible policies. Citizens are policy-motivated in that their utility depends on the policy which is implemented. A citizen ℓ's preference ordering over the set of alternatives $\mathcal{A}$ is represented by a utility function $u^\ell : \mathcal{A} \rightarrow \Re$. Each citizen ℓ is assumed to have a unique ideal policy w_ℓ, where $w_\ell \equiv \text{argmax}_{w \in \mathcal{A}} u^\ell(w)$.

There are three stages to the political process. At the first stage, each citizen chooses whether or not to enter the electoral race. In order to stand as a candidate, a citizen must incur the utility cost $\delta > 0$. At the second stage, each citizen decides which of the self-declared candidates to vote for. The election is held under AV. The candidate receiving the highest number of votes is elected. In the event of a tie

[11]Scholars have also studied the effect of different electoral systems on corruption (see Myerson (1993a) and Myerson (2000)) and the incentives to favor minority interests (see Myerson (1993b)).

between several candidates, each of them is elected with an equal probability. If no candidate enters the race, a default outcome $x_0 \in \mathcal{A}$ is implemented. At the third stage, the elected candidate chooses the policy to be implemented. We now analyze each of these stages in reverse order.

3.1. **Policy selection stage.** A key feature of the citizen-candidate model is that the citizens are policy motivated and cannot commit to implementing any policy other than their preferred policy. As a result, any candidate j who wins the election must implement his preferred policy w_j.

Let v_j^ℓ be citizen ℓ's payoff when citizen j is in office, with $v_j^\ell \equiv u^\ell (w_j)$. Also, denote by v_0^ℓ the payoff received by ℓ when no citizen runs for office and the default outcome is implemented.

3.2. **Voting stage.** Let $\mathcal{C} \subseteq \mathcal{N}$ denote a non-empty set of candidates who are running for office. Under AV, each citizen can vote for as many candidates as he wants, with the restriction that at most one indivisible vote can be cast per candidate. We describe citizen ℓ's voting decision by $\alpha^\ell (\mathcal{C}) \in \{0,1\}^{|\mathcal{C}|}$, whereby $\alpha_j^\ell = 1$ (0) means that citizen ℓ votes (does not vote) for candidate j. Let $\alpha(\mathcal{C}) = \left\{\alpha^1 (\mathcal{C}), ..., \alpha^{|\mathcal{N}|} (\mathcal{C})\right\}$ denote the profile of voting decisions.[12]

We shall call the set of candidates who receive the most votes as the *Winning Set* and denote it by $W(\mathcal{C}, \alpha)$. Formally,

$$W(\mathcal{C}, \alpha) \equiv \left\{ j \in \mathcal{C} : \sum_{\ell \in \mathcal{N}} \alpha_j^\ell \geq \sum_{\ell \in \mathcal{N}} \alpha_k^\ell, \text{ for all } k \in \mathcal{C} \right\}$$

Let $p_j(\mathcal{C}, \alpha)$ denote the probability that a candidate $j \in \mathcal{C}$ becomes the policy-maker given the voting profile α. Hence, $p_j(\mathcal{C}, \alpha) = \frac{1}{|W(\mathcal{C}, \alpha)|}$ if $j \in W(\mathcal{C}, \alpha)$ and 0 otherwise. We can now define a (pure strategy) voting equilibrium.

[12]Whenever it is possible to do so without causing a confusion, we shall omit $\mathcal{C}$ and denote the voting decisions and voting profile as α^ℓ and α, respectively.

Definition 1. (**Voting Equilibrium**) *Given a non-empty set of candidates* $\mathcal{C}$*, a strategy profile* $\alpha^*(\mathcal{C})$*is an equilibrium of the voting stage if for any citizen* $\ell \in \mathcal{N}$*,*

(i) $\sum_{j\in\mathcal{C}} p_j\left(\mathcal{C};\alpha^{\ell*},\alpha^{-\ell*}\right)\ v_j^\ell \geq \sum_{j\in\mathcal{C}} p_j\left(\mathcal{C};\alpha^{\ell},\alpha^{-\ell*}\right)\ v_j^\ell$*, for all* $\alpha^\ell \in \{0,1\}^{|\mathcal{C}|}$*, and*

(ii) $\alpha^{\ell*}$ *is weakly undominated.*

The first condition means that each citizen chooses his voting strategy in order to maximize his expected utility, taking into account others' voting strategies and correctly anticipating the policy implemented by the winner. The second condition is a standard refinement used in the voting literature. Under the plurality rule, this condition means that a voter does not vote for any of the least preferred alternatives. Under AV, it means that a voter does not vote for any of the least preferred alternatives *and* that he votes for all of the most preferred alternatives. We state this formally in lemma 1 (due to Brams and Fishburn (1978)).

To simplify our subsequent exposition, let us introduce the following notation. Consider a non-empty set of candidates- $\mathcal{C}$. For any $\ell \in \mathcal{N}$, let $G^\ell(\mathcal{C}) \equiv \left\{j \in \mathcal{C} : v_j^\ell \geq v_k^\ell \text{ for all } k \in \mathcal{C}\right\}$ denote the set of citizen ℓ's most preferred candidates and similarly, let $L^\ell(\mathcal{C}) \equiv \left\{j \in \mathcal{C} : v_k^\ell \geq v_j^\ell \text{ for all } k \in \mathcal{C}\right\}$ denote the set of citizen ℓ's least preferred candidates. The following lemma characterizes the set of weakly undominated voting strategies.

Lemma 1. *A voting strategy is weakly undominated for citizen* ℓ *if and only if* $j \in G^\ell(\mathcal{C}) \Rightarrow \alpha_j^\ell = 1$ *and* $j \in L^\ell(\mathcal{C}) \Rightarrow \alpha_j^\ell = 0$*.*

3.3. **Entry stage.** Each citizen must decide whether or not to stand for election at a utility cost $\delta > 0$. Let $s^\ell \in \{0,1\}$ be citizen ℓ's entry decision where $s^\ell = 1$ (0, resp.) means that citizen ℓ chooses to stand (not to stand, resp.) for election. The pure-strategy profile is denoted by $s = \left\{s^1, ..., s^{|\mathcal{N}|}\right\}$ and the set of candidates associated with it by $\mathcal{C}(s) \equiv \left\{\ell \in \mathcal{N} : s^\ell = 1\right\}$. Citizens correctly anticipate others' entry strategies as well as the voting profile associated with each set of candidates and then make their entry decisions strategically. Citizen ℓ's expected payoff for a given strategy profile s is thus

$$U^{\ell}\left(\mathcal{C}\left(s\right),\alpha\right)=\sum_{j\in\mathcal{N}\cup\{0\}}p_{j}\left(\mathcal{C}\left(s\right),\alpha\left(\mathcal{C}\left(s\right)\right)\right)\ v_{j}^{\ell}-s^{\ell}\ \delta$$

where $p_0\left(.\right)$ denotes the probability that the set of candidates is empty.

A pure-strategy equilibrium of the entry stage is a strategy profile s^* such that for each citizen ℓ, $s^{\ell*}$ is a best-response to $s^{-\ell*}$. However, an entry stage equilibrium in pure strategies need not exist. Hence we may have to consider mixed strategies. Let $\sigma^{\ell}\in[0,1]$ denote the mixed strategy of agent ℓ, where σ^{ℓ} should be interpreted as the probability that ℓ enters the political race. A mixed strategy equilibrium of the entry stage σ^* can be defined analogously.

3.4. **Political equilibrium.** We can now define a political equilibrium as a pair $\left(\sigma^*,\alpha^*\left(\cdot\right)\right)$ consisting of an entry decision profile σ^* and a voting strategy profile $\alpha^*\left(\cdot\right)$ such that: (i) $\alpha^*\left(\cdot\right)$ is a voting equilibrium for any non-empty set of self-declared candidates $\mathcal{C}$; and (ii) σ^* is an equilibrium of the entry game given the voting function $\alpha^*\left(\cdot\right)$.

It is easy to show the existence of voting equilibria in pure strategies. Given the finiteness of the game we know that there exists an equilibrium of the entry stage game. Hence we get

Proposition 1. *A political equilibrium exists.*

3.5. **Refining the voting behavior.** As we saw in lemma 1, the standard voting refinement of weak non-dominance puts very little restriction on the permissible voting behavior. Not only does this mean that the notion of Nash equilibrium gives us little predictive power, it also fails to capture the intuitive (and empirically observed) way in which people vote under AV. Hence, we consider two plausible notions of sincere voting behavior: *sincere voting strategies* and *relatively sincere voting strategies*. We then proceed to examine the sense in which these notions are consistent with the strategic calculus of voting. But first we need to introduce some notation.

Let $BR^{\ell}(\mathcal{C},\alpha) \equiv \arg\max_{\alpha^{\ell}\in\{0,1\}^{|\mathcal{C}|}} U^{\ell}\left(\mathcal{C};\alpha^{\ell},\alpha^{-\ell}\right)$ be citizen ℓ's set of best-responses to the strategy profile $\alpha^{-\ell}$. $BR^{\ell}(\mathcal{C},\alpha)$ consists of the (pure) voting strategies that maximize citizen ℓ's expected utility given others' voting strategies and a set of candidates $\mathcal{C}$. Now we are in a position to formally define our various notions of sincerity.

Definition 2. (**Sincerity**) *Let* $|\mathcal{C}| \geq 2$. *A citizen ℓ's voting strategy is said to be sincere if for all j and $k \in \mathcal{C}$, with $j \neq k$ and $v_k^{\ell} > v_j^{\ell}$, the following two conditions are satisfied: (i) $\alpha_k^{\ell} = 1$ whenever $\alpha_j^{\ell} = 1$; and (ii) $\alpha_j^{\ell} = 0$ whenever $\alpha_k^{\ell} = 0$.*

This concept requires that if a citizen votes for a candidate j, then he must also cast a vote for every candidate k whom he strictly prefers over j. Similarly, if he does not vote for a candidate k, then he must not cast a vote for any candidate j who gives him a strictly lower utility compared to k. In any election with no more than three candidates, every weakly undominated voting strategy satisfies this definition.

However, for a large number of candidates, the concept of sincerity does not put any restriction on how many candidates a voter should vote for. He could vote for only the members of $G^{\ell}(\mathcal{C})$ or he could vote for everyone outside of $L^{\ell}(\mathcal{C})$ or he could draw a cutoff at any intermediate candidate. The concept of Relative Sincerity provides a condition on where such a cutoff should be drawn. It requires a citizen to vote (not to vote, resp.) for all the candidates who give her a strictly higher (lower, resp.) utility than the lottery over the winners.[13] Let us define the notion of relative sincerity formally,

Definition 3. (**Relative Sincerity**) *Let* $|\mathcal{C}| \geq 2$. *A citizen ℓ's voting strategy α^{ℓ} is said to be sincere relative to $\alpha^{-\ell}$ if, whenever $G^{\ell}(\mathcal{C}) \neq L^{\ell}(\mathcal{C})$, the following two conditions hold: (i) $\alpha_i^{\ell} = 1$ whenever $v_i^{\ell} > U^{\ell}\left(\mathcal{C};\alpha^{\ell},\alpha^{-\ell}\right)$; and (ii) $\alpha_i^{\ell} = 0$ whenever $v_i^{\ell} < U^{\ell}\left(\mathcal{C};\alpha^{\ell},\alpha^{-\ell}\right)$.*

[13]See Merrill and Nagel (1987) for empirical evidence supporting the validity of this notion.

The following lemma shows that the equilibrium voting strategies *must be* relatively sincere on the set of serious contenders.

Lemma 2. *Suppose that* $|W(\mathcal{C}, \alpha(.))| \geq 2$ *and* $G^{\ell}(W) \neq L^{\ell}(W)$. *Then for all* $i \in W$ *and* $\ell \in \mathcal{N}$, *we have for any voting profile* $\alpha^{\ell} \in BR^{\ell}(\mathcal{C}, \alpha)$,

(*i*) *if* $\alpha_i^{\ell} = 1$, *then* $v_i^{\ell} \geq \frac{1}{|W|} \sum_{j \in W} v_j^{\ell}$, *and*

(*ii*) *if* $v_i^{\ell} > \frac{1}{|W|} \sum_{j \in W} v_j^{\ell}$, *then* $\alpha_i^{\ell} = 1$.

This lemma provides a simple characterization of agents' voting strategies over the winning set. It states that in the event of a tie, a citizen who is not indifferent between all serious contenders should vote (resp. not vote) for all those who give her a utility strictly higher (resp. lower) than the lottery over those candidates. Note that this lemma implies that if *all* candidates are serious contenders, then the equilibrium voting behavior necessarily satisfies relative sincerity.

Our next result shows that the notion of relative sincerity is consistent with the strategic calculus of voting in the sense that each voter's best-response set contains such a strategy.

Proposition 2. *Let* $|\mathcal{C}| \geq 2$. *For any* $\ell \in \mathcal{N}$ *and strategy profile* $\alpha^{-\ell}$, *there exists a strategy* $\alpha^{\ell} \in BR^{\ell}(\mathcal{C}, \alpha)$ *such that* α^{ℓ} *is sincere relative to* $\alpha^{-\ell}$.

Since the concept of relative sincerity is stronger than the concept of sincerity, proposition 2 implies that sincere strategies are contained in the agents' best response sets as well.

In our next section we prove the existence of equilibria in relatively sincere strategies in the context of a unidimensional model with single peaked preferences. Then we explore how this refinement helps us to characterize the plausible electoral equilibria under AV.

4. Unidimensional Policy Space

We now analyze a special case of the model developed in section 3 - one with a unidimensional policy space and distance preferences. There are two motivations for doing so. First, since the previous literature on AV has focused, almost exclusively, on the unidimensional model, the following analysis highlights the important differences between our approach vis-a-vis the previous ones. In particular, we show that there are important implications of endogenizing the set of candidates. Second, we are able to compare the equilibria under AV with those under the plurality rule and examine the effect of AV on policy moderation.

Consider the set of policy alternatives to be $[0, 1]$. Citizens' preferences are represented by a utility function $u^i(w)$ which satisfies the following properties:

- (i) (Single-peakedness) Given any two policies x and y such that $i \leq x \leq y$ or $y \leq x \leq i$, we must have $u^i(x) \geq u^i(y)$.
- (ii) (Concavity) Let D be a set of policies and let $a_D \equiv \sum_{d \in D} d$ be the average policy. Then, for each i, $u^i(a_D) \geq \frac{1}{|D|} \sum_{d \in D} u^i(d)$.
- (iii) (Distance Condition) Consider any citizens i and j and any policies x and y, such that $i \leq j$ and $x \leq y$, then $u^i(y) \geq u^i(x) \Rightarrow u^j(y) \geq u^j(x)$.

Each citizen ℓ has a unique ideal policy $w_\ell \in [0, 1]$ though more than one citizens could have the same ideal policy. We normalize $u^\ell(w_\ell)$ to 0 and denote by v_j^ℓ citizen ℓ's utility from policy w_j. Finally, let m denote the median voter's ideal policy and let $\mathcal{M} \equiv \{\ell \in \mathcal{N} : w_\ell = m\}$ be the set of citizens at that position. We also assume $\frac{|\mathcal{N} \setminus \mathcal{M}|}{2}$ to be an integer. To simplify the analysis, we assume that when indifferent between voting and not voting over a subset of candidates in $\mathcal{C}$, a voter votes for all of them. Now we proceed to characterize the set of political equilibria under AV.

4.1. **Characterization of Political Equilibria.** Our first result concerns the number of winning positions in any voting equilibrium. We prove that in any voting equilibrium, irrespective of the number of candidates standing for election, there can be at most two winning positions. Furthermore, when there are two distinct winning positions, they must be such that the median voter is indifferent between the two.

Proposition 3. $|W(\mathcal{C}, \alpha(\cdot))| \geq 2$ *if and only if for all* $i \in W(\mathcal{C}, \alpha(\cdot))$, $w_i \in \{w_1, w_2\}$ *with* $0 \leq w_1 < m < w_2 \leq 1$ *and* $v_1^m = v_2^m$.

To understand the intuition behind proposition 3 consider the case of a tie between a left moderate, a right moderate and a left extremist. Also, suppose that the average policy lies between the two moderate positions. We know from lemma 2 (Relative Sincerity on the Winning Set) that the the left moderate must receive the votes of all the voters to the left of him. Also, if a citizen to the right of the left moderate candidate is voting for the left extremist, he must be voting for the left moderate too. Hence the if the two left candidates were to tie, the same set of voters must be voting for them! By the same principle, each voter must be voting either for all the candidates left of the average or all the candidates right of the average (but not both). But then, by the distance property, if the median voter votes for the candidates on the left of the average policy, then he everyone to the left of the median must be doing the same, and therefore there could not have been a tie between the policies on the left of the average and those on the right.

The above result is interesting because it answers peoples' concern that AV may obliterate the two party system and lead to a proliferation of multiple platforms at which winning candidate can be found. Our result shows that this is indeed not the case. In fact, in any political equilibria without spoiler candidates, there will be at most two positions at which candidates will enter. Hence, AV satisfies a variant of the Duverger's law. Duverger (1954) observed that the plurality rule favors a two party system since the voters', fearing that they might waste their vote, will choose not to vote for a potential entrant. Our analysis shows that a similar result emerged under AV as well.

There are, however, two important differences. The first one concerns the number of candidates. While under the plurality rule, there will be only two serious candidates, AV does not prevent the entry of more than one candidate at each winning position. Indeed, *even when* citizens are exclusively policy-motivated, several may decide to run on the same platform in order to increase the probability that their ideal policy is implemented. Hence, there can be more than two serious contenders but only two policy outcomes. The second difference concerns the logic driving the result about two winning positions . Under AV, the result comes from the relative sincerity of the voting behavior on the winning set. Under the plurality rule, it comes from the 'wasting-the-vote' argument. Indeed, voters' tendency to ignore a third party is based on a self-fulfilling prophecy that a vote for this party would be wasted.

We are now in a position to characterize the set of 'serious political equilibria' (i.e. those involving no spoiler candidates). We know that there are two classes of such equilibria- one position equilibria and two positions equilibria. The following proposition characterizes the set of one position equilibria

Proposition 4. *There exists a political equilibrium with citizen i running unopposed if and only if*

(*i*) $-v_0^i \geq \delta$, *and*

(*ii*) $\nexists\ h \in \mathcal{N} \setminus \{i\}$ *such that either* $v_h^m > v_i^m$ *and* $-v_i^h \geq \delta$, *or* $v_h^m = v_i^m$ *and* $-v_i^h \geq 2\ \delta$.

The above conditions are extremely intuitive. The first condition requires that the candidate prefers to run against the *status quo*. The second condition implies that no other candidate prefers to enter the race. One immediate implication of condition (*ii*) is that in a one-position equilibrium, the is exactly one candidate running at that position. Moreover, the set of outcomes attained under the one position equilibria under AV are equivalent to those attained under the one candidate equilibria under the plurality rule. We shall prove this result in a more general setting in the next section.

We now characterize a two position, serious political equilibrium. We know from proposition 3 that the two positions must be on two sides of the median. It must also be true that the candidates who are running are worse off quitting and those who are not running will either not get sufficient votes or the increment in their payoff does not justify the cost of running. Formally,

Proposition 5. *Consider a two position, serious political equilibrium with* $|\mathcal{C}| \equiv c \geq 2$*, then the following conditions must be satisfied*

(*i*) *there exist two positions* w_L *and* w_R *such that,* $0 \leq w_L < m < w_R \leq 1$*, and* $v_L^m = v_R^m$.

(*ii*) *let* $c(R)$ *denote the number of number of candidates running at position* w_R*, then for every* $i \in \mathcal{C}$ *running at* w_L*,* $-\frac{c_R}{c(c-1)} v_R^i \geq \delta$ *and a similar condition holds for candidates at* w_R.

(*iii*) *for any* $i \notin \mathcal{C}$ *with ideal policy* w_L*,* $-\frac{c_R}{c(c-1)} v_R^i < \delta$ *and a similar condition holds for any* $i \notin \mathcal{C}$ *with the ideal policy at* w_R.

(*iv*) *there does not exist a citizen* k *with* $w_k \in (w_L, w_R)$ *such that either*

(*iva*) $\left|\left\{\ell \in \mathcal{N} : k \in G^\ell\left(\mathcal{C} \cup \{k\}\right)\right\}\right| > \frac{|\mathcal{N} \backslash \mathcal{M}|}{2}$ *and* $-\frac{1}{c} \sum_{i \in \{1,2\}} c_i \, v_i^k \geq \delta$*, or*

(*ivb*) $\left|\left\{\ell \in \mathcal{N} : k \in G^\ell\left(\mathcal{C} \cup \{k\}\right)\right\}\right| = \frac{|\mathcal{N} \backslash \mathcal{M}|}{2}$ *and* $-\frac{1}{c\ (c+1)} \sum_{i \in \{1,2\}} c_i \, v_i^k \geq \delta$.

Moreover, the above conditions are sufficient if for all $k \in \mathcal{N}$ *with* $w_k \in (w_1, w_2)$*,* $\left|\left\{\ell \in \mathcal{N} : k \in G^\ell\left(\mathcal{C} \cup \{k\}\right)\right\}\right| < \frac{|\mathcal{N} \backslash \mathcal{M}|}{2} - 1$.

There are two important ways in which the two positions equilibria under AV differ from those under the plurality rule. First, unlike under the plurality rule, there may be more than one candidates running at the same position. Under the plurality rule, two or more candidates running at the same position *hurt* one another by splitting the votes. In contrast, under AV, they *help* each other by increasing the probability with which their position is chosen. Second, under the plurality rule, candidates face no risk of entry from other candidates, even if these candidates are more centrist. Voters may not want to vote for the centrist candidate because that will reduce the vote share of their preferred candidate, leading to a victory to their least preferred candidate. Such a problem does not arise under AV and hence, a centrist candidate may find it worthwhile to enter (and win) if condition (*iv*) in the above proposition is violated.

4.2. **A Comparison between AV and the Plurality Rule.** We now investigate the difference between the sets of non-spoiler political equilibria under AV and the plurality rule. Since our main focus is on which policy is implemented, we will consider a political equilibrium under the plurality rule as equivalent to one under AV if they both generate the same set of policy outcomes.

Before proceeding, we need to characterize equilibria under the plurality rule. Since our framework is similar to the one considered in Besley and Coate (1997), we can compare our outcomes under AV with their results on the political equilibria under the plurality rule. To facilitate the comparison, we need to modify their framework to allow for the possibility of more than one candidates sharing the same ideal position. After this modification to Besley and Coate (1997), it can be shown that such equilibria exhibit the following features: (*i*) the necessary and sufficient conditions for the existence of one candidate political equilibria under the plurality rule are identical to those in proposition 4, (*ii*) the first two conditions of proposition 5 are analogous to the necessary and sufficient conditions for a two-candidate political equilibrium under the plurality rule. Also the other conditions of proposition 5 are not necessary under the plurality rule. Indeed, under the plurality rule, votes would split between several candidates running on the same platform.[14] (*iii*) there does not exist a three-candidate political equilibrium; and (*iv*) in political equilibria with four or more candidates, only one or two are winning.

From those features, we can conclude that, the outcomes generated by the one candidate equilibria are identical under the two systems. Moreover, the set outcomes from the two position, serious equilibria under AV is a subset of those obtained under plurality rule. The following proposition goes further in characterizing the conditions under which the equilibrium set of policy outcomes under AV can be more moderate than the one arising under the plurality rule.

Proposition 6. *Consider the class of non-spoiler candidate political equilibria. The set of outcomes from the political equilibria under AV is a subset of the set of outcomes under the plurality rule. In addition, when preferences are symmetric, former set is more moderate than the latter in the sense that if the lottery that implements* w_1 *and* w_2 *is an equilibrium outcome under AV, then for every equilibrium under the plurality rule that implements* $\widetilde{w}_1$ *and* $\widetilde{w}_2$*, with* $[\widetilde{w}_1, \widetilde{w}_2] \subset [w_1, w_2]$*, there exists a political equilibrium under AV where the same lottery is the electoral outcome.*

To see the above proposition in action, consider the following example.

Example 1. Suppose there are 36 voters whose ideal points are distributed on the set $\{0,1,\ldots,9,10\}$. Let there be 6 voters with ideal point 5 and $6-k$ voters with ideal point $5 \pm k$. Voters have Euclidean Preferences, denoted by $v_j^{\ell} = \|w_{\ell} - w_j\|$[15]. Suppose that the cost of running for office is $\delta = 0.8$. We denote by $\{w_j, w_k\}$ an equilibrium under which candidates with ideal points w_j and w_k run against each other.

[14] In the limit, one candidate will get all the votes and the others none. But in that case, the latter candidates would be better-off exiting the race.

[15] We denote Euclidean preferences by $\|.\|$ in order to avoid any confusion with the cardinality sign $|.|$

What are the two-candidate equilibria under the plurality rule? It is easy to see that $\{0, 10\}, \{1, 9\}, \{2, 8\}, \{3, 7\}, \{4, 6\}$ are all two-candidate equilibria under the plurality rule. The reason for this result is that a moderate candidate will not want to run due to the belief that he may not receive any votes. This belief is consistent with equilibrium voting behavior since the voters do not want to waste their vote on the moderate candidate only to lead the other extremist to win. For instance, consider the scenario in which 0 and 10 are running against each other. Suppose that a moderate candidate, say at 5, is considering whether to enter the race. A voter on the left of 5 will not want to vote for the candidate at 5 because that will lead to the candidate at 10 winning outright!

Under AV, potential entrants do not face this problem. Suppose that 0 and 10 are running against each other and 5 entered. For everybody with ideal points at 3,4,5,6 and 7, voting for 5 is always a part of *any* weakly dominant strategy. This means that 5 will receive 24 votes while 0 and 10 will receive at most 15 votes. It follows that the equilibrium $\{0, 10\}$ is destroyed by the credible threat of entry by a candidate at 5! By a similar logic, one can show that only $\{3, 7\}$ and $\{4, 6\}$ survive as two-candidate equilibria under AV, as the rest are destroyed by a credible threat of an entry by a moderate.

However, there are two qualifications to be kept in mind regarding proposition 6. First, the extent of moderation under AV need not be very substantial, and second, if preferences are asymmetric, then AV may not produce moderation. Our next two examples shed light on these qualifications.

Example 2. A community has to select a representative who, once elected, will have to choose the share of a fixed budget allocated to a public good. Let citizens' ideal policies $w_i \in \left\{0, \frac{1}{100}, \frac{2}{100}, ..., 1\right\}$ with five citizens of each type.[16] The median voter's ideal policy is $\frac{50}{100}$. Citizens have Euclidean preferences over the public good, $u^{\ell}(g) = -\|w_\ell - g\|$, and bear a cost $\delta = \frac{1}{20}$ if they decide to run for election.

First note that any pair of citizens, with $w_i + w_j = 1$ and $\|w_i - w_j\| \geq \frac{1}{10}$, running against each other is a political equilibrium under the plurality rule.

Now consider the citizens with ideal points at $\frac{3}{100}$ and $\frac{97}{100}$. It is easy to see that the first three necessary conditions of proposition 5 are satisfied when five citizens of each type are running for election. Also, for any citizen whose ideal point lies in-between, the set of citizens who would prefer her to the other candidates is at most $\frac{5\ (97-3)}{2} = 235$. In the same time, $\frac{|\mathcal{N} \setminus \mathcal{M}|}{2} - 1 = 249$. Hence, both the fourth and the sufficiency conditions are satisfied as well. We can then conclude that the lottery where $\frac{3}{100}$ and $\frac{97}{100}$ are each adopted with equal probabilities is an equilibrium policy outcome under AV. Furthermore, $\left\{\frac{4}{100}, \frac{96}{100}\right\}$, $\left\{\frac{5}{100}, \frac{95}{100}\right\}$, ... and $\left\{\frac{45}{100}, \frac{55}{100}\right\}$ are equilibria as well.

Example 3. Suppose preferences are now represented by the following utility function

$$\begin{aligned} u^{\ell}(w_i) &= -\frac{1}{2}\ \|w_\ell - w_i\| \quad \text{if } w_\ell \geq w_i, \\ u^{\ell}(w_i) &= -2\ \|w_\ell - w_i\| \quad \text{if } w_i \geq w_\ell. \end{aligned}$$

Then, under Plurality voting, the set of equilibrium lotteries over the policies is $\left\{\left\{\frac{2}{100}, \frac{62}{100}\right\}, \left\{\frac{6}{100}, \frac{61}{100}\right\}, ..., \left\{\frac{34}{100}, \frac{54}{100}\right\}\right\}$, where, in each of them, both policies are adopted with equal probabilities.

Consider the equilibrium where, under the plurality rule, $\frac{34}{100}$ and $\frac{54}{100}$ are adopted with equal probabilities. It turns out that this lottery cannot be supported as a political equilibrium outcome under AV. The intuition is as follows: Suppose that

[16] Note that the argument does not depend on the restriction that there are five citizens of each type. The rationale for this assumption will become clear when we will consider the second qualification.

there is one citizen running at $\frac{34}{100}$ and another one at $\frac{54}{100}$.[17] Then, a second candidate will want to stand on the leftist platform (since his expected benefit of running, net of the entry cost, is equal to $\frac{1}{60}$.) At the same time, no other citizen at the rightist position will want to enter. This asymmetry comes from the difference in preferences. As a result, $\frac{34}{100}$ will be implemented with probability two thirds and $\frac{54}{100}$ with probability one third. This means that the benefit of running for the rightist candidate will now be $\frac{1}{30}$, less than the entry cost. Anticipating this, the rightist candidate will then choose not to enter in the first place. Thus, $\left\{\frac{34}{100}, \frac{54}{100}\right\}$ cannot be supported by a political equilibrium under AV. The same is true for $\left\{\frac{30}{100}, \frac{55}{100}\right\}$ and $\left\{\frac{22}{100}, \frac{57}{100}\right\}$.

Moreover, in the political equilibria where the other pairs are implemented, the leftist policy (and thus the most extreme one) is adopted with a higher probability than the rightist one. For example, $\left\{\frac{2}{100}, \frac{62}{100}\right\}$ is supported by a political equilibrium under AV as well, but with the set of candidates and the winning set both consisting of four candidates with $\frac{2}{100}$ as an ideal policy and one with $\frac{62}{100}$.

4.3. **Refining the Voting Behavior.** In section 3 we developed an intuitively plausible notion of voting behavior, viz., relative sincerity. Our analysis showed that relatively sincere strategies are compatible with rational behavior in the sense that players' best response sets always contain such strategies (see proposition 2). Our next result goes further to prove the existence of equilibria in relatively sincere strategies in case of the unidimensional model.[18]

Lemma 3. *If the policy space is unidimensional and the preferences satisfy single-peakedness and the distance property, then for any candidate set $\mathcal{C} \neq \varnothing$, there exists a voting equilibrium in relatively sincere strategies.*

[17] Two or more candidates at each position would violate condition (*ii*) of proposition 5.

[18] We believe that the existence result holds in general as well, but the proof eludes us for the time being.

This sub-section explores the implications of relatively sincere voting behavior on the degree of moderation attained under AV. Our next proposition shows that if voters play relatively sincere voting strategies, then the equilibrium outcome of any serious political equilibrium must be 'close' to the median voter's ideal policy.

Proposition 7. *Suppose that the voting behavior satisfies Relative Sincerity and* $-v^m_{x_0} \geq \delta$. *In any political equilibrium with* $W(\mathcal{C}, \alpha(.)) = \mathcal{C}$, $-v^m_i < \delta$ *for all* $i \in \mathcal{C}$.

To understand the logic behind the above result, note that there are two types of serious equilibria- those with one winning position and those with two winning positions. It follows immediately from proposition 4 that the outcome under the first case cannot be 'too far' from the median. The second, and more interesting, case is that of two winning positions. We know from proposition 3, that the two winning positions must be distributed symmetrically from the median position. However, by concavity of preferences, the median position must be preferred by *at least* a majority of voters. This means that if voters voted relatively sincerely, a candidate who entering at the median position will win outright. Hence, the only two position equilibria that survive are those which generate outcomes close to the median voter's ideal policy.

To summarize, our analysis of the one dimensional model shows that AV leads to moderate outcomes if we look at equilibria without spoiler candidates. Moreover, the extent of moderation is substantial if we consider equilibria in relatively sincere strategies.

4.4. **The case of Spoiler Candidates.** We have primarily focused our attention on the case of serious candidates. There is another class of equilibria- those involving spoiler candidates. Under such equilibria there are candidates running in the race, even though they who do not stand a chance of winning. These equilibria, though theoretically possible, rely heavily on beliefs on part of the spoilers which are unreasonable. For instance,

Example 4. Consider the preferences and population as described in example 1. $\mathcal{C} = \{0, 1, 9, 10\}$ and $W(\mathcal{C}, \alpha(.)) = \{1, 9\}$ can be sustained as a four-candidate political equilibrium in the above example. For instance, 0 believes that his exit will cause everyone to the left of 9 to vote for 9. This will lead to 9 winning outright. Similarly, 10 believes that his exit will 1 to win outright. Moreover, every potential entrant believes that his entry would trigger voting behavior that would lead to either 1 or 9 getting elected outright. This could deter the potential entrants from running. Thus the above behavior constitutes an equilibrium.

5. Some Results in the General Policy Space

In this section we extend our analysis to the case of general preferences and policy spaces. One advantage of our approach vis-a-vis the Downsian approach is our ability to obtain equilibrium predictions in such a general setting. We saw in the earlier section that the set of possible candidates running under AV can be a subset of those running under the plurality rule. As our next example shows, there are outcomes which can arise as political equilibria under AV but cannot arise as outcomes under the plurality rule.

Example 5. A community must elect a local official who is in charge of undertaking a public investment. The public investment project can be undertaken on three possible scales: small, medium or large. If nobody runs for office, no investment is undertaken and citizens get payoff 0. The community is divided into three types of citizens whose preferences over the scale of the project are presented below.

	Investment Project		
	small	medium	large
Group 1	12	7	0
Group 2	7	12	0
Group 3	7	7	12

Let n_i denote the number of type i citizens. We assume that $n_1 = n_2 > 2n_3$.

Note that a type-1 running against a type-2 cannot be an equilibrium configuration under the plurality rule since a type-3 citizen will now find it beneficial to stand as a candidate. Indeed, if one type-3 citizen also runs, he will get the votes of his fellows (since they will not abstain given that $v_3^3 > v_1^3 = v_2^3$ when in the same time admissibility will require $\alpha_1^3 = \alpha_2^3 = 0$) while the best that type-1 and type-2 citizens can do is all to vote for one of the other two candidates. So the type-3 candidate will either win outright or tie with one of the other candidates. He will then get an expected utility of at least $\frac{1}{2}(7+12)-2 = \frac{15}{2}$ (which occurs if there is a tie) compared to the 7 he would get by not entering. Hence he is indeed better-off standing for election.

However, there exists a voting strategy profile under AV that sustains a type-1 running against a type-2 as an equilibrium. If the set of candidates consists of citizens of types 1 and 2, the only admissible (and equilibrium) voting strategy is the type-1 and type-2 citizens voting for their candidate and the type-3 citizens abstaining.

Suppose now that a citizen of type 3 enters the race and let $\alpha^h(\{1,2,3\}) = \{1,2\}$ for $h \in \{1,2\}$ and $\alpha^3(\{1,2,3\}) = \{3\}$. This candidate then gets the votes of all his fellows while the other citizens cast a vote for each of the other two. All three contenders thus tie. The type-3 candidate would then get an expected utility of $\frac{20}{3}$, less than the 7 he gets by not running. This means that he has no incentive to stand for election. Now we need to check that this voting function is indeed an equilibrium one, i.e. no citizen would gain by changing her vote. For a type-3 citizen, this is the unique admissible voting strategy. A type-1 citizen would get a lower expected utility if she decides not to vote for the type-2 candidate ($\frac{12+0}{2} = 6$ *vs* $\frac{12+7+0}{3} = \frac{19}{3}$) and *a fortiori* for her fellow candidate. Hence, she has no incentive to deviate. The same is true for the citizens of type 2.

It remains to show that neither of the two candidates wants to exit and that no other citizen wants to enter the electoral race. If the type-1 candidate decides to step out, he will get a utility equal to 7 (since the type-2 candidate is then going to win outright), lower than the $\frac{15}{2}$ he was getting. Hence, he has no incentive to exit the race. The same is true for the type-2 candidate.

If a citizen of type $h \in \{1, 2\}$ decides to enter, he will tie with the other two candidates. His expected utility will then be equal to $\frac{25}{3}$, less than the $\frac{19}{2}$ he was getting. So he will not enter. And we have shown above that a type-3 citizen will not want to stand for election.

Hence, there exists under AV, but not under the plurality rule, a political equilibrium where one citizen of type 1 and one of type 2 are running against each other. Moreover, this equilibrium satisfies relative sincerity.

From proposition 6 and example 5 we can conclude that

Proposition 8. *In general, there is no equivalence between the set of candidates running under AV and the plurality rule.*

This proposition highlights the fact that different electoral rules create different incentives for candidates to enter the political race. There have been several attempts to compare electoral rules by 'reconstructing' the outcomes of an election under various voting rules. The above proposition cautions us such attempts may be misleading since electoral rules make a difference at the candidate entry stage as well as at the voting stage.

Endogenizing candidate entry also calls for a re-examination of the desirability properties of voting systems which are based on an analysis over exogenous set of alternatives. There are two Condorcet criteria which have received attention in the literature. A voting system is said to satisfy the Condorcet Winner Criterion if it has a voting equilibrium that elects a Condorcet winner, i.e., an alternative that beats any other alternative in a pairwise majority vote. Similarly, a voting system is said to satisfy the Condorcet Loser Criterion if it does not have an voting equilibrium that elects a Condorcet loser, i.e., the alternative that is defeated by every other alternative in a pairwise majority vote.

One of the criticisms of the plurality rule is that it fails to satisfy the Condorcet criteria. In fact, it may even elect a Condorcet loser while a Condorcet winner exists. Our next example shows that AV is susceptible to the very same problems.

Example 6. Consider again an economy which has to decide on a public investment. There are three projects which differ in terms of their size, i.e. there is a small project, a medium one and a large one. There are four preference profiles in the community. Utility levels are

	Project size		
	small	medium	large
Type 1	10	6	2
Type 2	2	10	2
Type 3	2	6	10
Type 4	10	0	0

Suppose that the distribution of citizens between these four groups is such that $n_1 + n_4 > n_2 + n_3 > n_4 + 1$ and $n_3 > n_1 + n_2$, with at least one citizen of each type. Also let the entry cost $\delta \in (1, 4)$.

It is easy to see that the small project is the Condorcet winner while the medium one is the Condorcet loser.

Now consider the situation where one citizen of each of the first three types are running against each other, i.e. $\mathcal{C} = \{1, 2, 3\}$. If elected the type-1 candidate will implement the small project, the type-2 candidate the medium one and the type-3 the large one. Hence, the Condorcet winner is the type-1 candidate and the Condorcet loser the one of type 2. Now, there exists an equilibrium under AV where the type-2 candidate is the only serious contender, the other two being spoiler candidates. This equilibrium is supported by the following voting behavior: (i) type-1 citizens vote for both the type-1 and type-2 candidates; (ii) type-2 citizens vote for their fellow candidate; (*iii*) type-3 citizens vote for both the type-2 and type-3 candidates; and (*iv*) type-4 citizens vote for the type-1 candidate.

There are several things to note about this example. First, it is also an equilibrium under the plurality rule where the type-2 candidate receives the votes of all citizens except the ones of type-4 citizens who cast their vote for the type-1 candidate. Second, this equilibrium also satisfies relative sincerity. Hence, even imposing the sincerity refinement on the voting behavior does not guarantee that AV satisfies the Condorcet criteria. Finally, note that from a Rawlsian point of view, the medium size project is the least-desirable outcome while the small project is the most-desirable one. The same is true from a utilitarian point of view if $2\ n_1 + 5\ n_4 > 4\ n_2 + 2\ n_3$ and $n_3 > n_1 + 2\ n_2$.[19]

It has been argued that when the preference profile of every citizen is dichotomous, Approval voting is the only one among non-rank scoring rules to satisfy the Condorcet winner criterion. However, this result no longer holds when the set of candidates is not fixed, as the next example shows.

Example 7. Consider again an economy which has to choose the level of a public investment among three projects which differ in terms of their size. The default outcome is the medium size project. There are three types of preference profiles in the community, each being dichotomous, with the following utility levels

	Project size		
	small	medium	large
Type 1	10	2	2
Type 2	9	10	9
Type 3	2	2	10

Let the entry cost δ be equal to two and suppose there are as many citizens of type 1 as of type 3 and less than of type 2 (i.e. $n_1 = n_3 < n_2$.) Hence, the Condorcet

[19]For instance, $n_1 = n_4 = 6$, $n_2 = 1$ and $n_3 = 9$ is one such distribution.

winner is the medium size project. However, the only equilibrium in pure strategies is such that one citizen of type 1 and one of type 3 are running against each other and tie. The unique admissible voting strategy is then such that each candidate gets the votes of his fellows while the type-2 citizens, being indifferent between both candidates, choose to abstain. Also, no other type-1 or type-3 citizen wants to stand for office. And a type-2 citizen does not want to enter the race since the entry cost is larger than the utility gain from implementing her preferred policy. As a result, the Condorcet winning alternative is never carried out, the type-1 (resp. type-3) candidate implementing the small (resp. large) project if elected.

6. Conclusion

In this paper we developed a model of political competition that enabled us to study electoral process under AV and to compare it with the plurality rule. While the existing studies of alternative electoral systems have focused on a fixed set of alternatives, we adopted the citizen-candidate framework to endogenize candidate entry.

We first examined the notion that AV encourages sincere voting behavior. To this end, we developed a refinement of voting behavior- namely, relative sincerity, which is consistent with the intuitive notion of sincere voting behavior under AV. We showed that relatively sincere voting behavior is consistent with the rational calculus of voting.

We then developed a one dimensional model of political competition with distance preferences. This set up enabled us to examine the claim that AV leads to more centrist policies as compared to the plurality rule. We found that the outcomes under AV are considerably more moderate than those that may arise under the plurality rule if we focus our attention on 'serious' equilibria in relatively sincere strategies. Hence our analysis found precise conditions, viz. no spoiler candidates and relatively sincere voting behavior, under which AV performs better than the plurality rule. However, we also find that there need be little policy moderation if these two conditions did not hold. Hence, we are more cautious in our support of AV over the plurality rule until we have empirical evidence on how prevalent these two conditions are. We also showed that when the policy space is unidimensional, there may be numerous candidates running for office, but all the serious candidates are clustered at no more than two positions! This is contrary to the popular intuition that AV may lead to a large number and variety of candidates.

We also were able to highlight the methodological contribution of the citizen-candidate approach to comparing electoral rules. We showed that in general there is no equivalence between the set of candidates running for election under AV and the plurality rule. Moreover, various properties of voting rules which are based on an analysis over a fixed set of alternatives may not hold when we allow for endogenous entry.

There are two possible extension of this work. On the empirical side, we would like to examine whether AV satisfies the conditions that we showed are sufficient for policy moderation. On the theoretical side, we would like to expand the citizen-candidate framework to studying other voting rules such as the Borda Rule, negative voting and so on.

Appendix

Proof. (**Lemma 1**)
(sufficiency) Let α^ℓ be a non-admissible strategy. Then we have to show that there exists another strategy $\widehat{\alpha}^\ell$ such that for all $\alpha^{-\ell}$, $U^\ell\left(\mathcal{C};\widehat{\alpha}^\ell,\alpha^{-\ell}\right) \geq U^\ell\left(\mathcal{C};\alpha^\ell,\alpha^{-\ell}\right)$ with a strict inequality for some $\alpha^{-\ell}$. But before proceeding, let $R(\mathcal{C};\alpha) \equiv \{i \in \mathcal{C} : \left(\sum_{\ell\in\mathcal{N}}\alpha_i^\ell\right)+1 = \sum_{\ell\in\mathcal{N}}\alpha_k^\ell \text{ for all } k \in W(\mathcal{C};\alpha)\}$ be the set of candidates who are one vote short to tie for election. There are two cases to consider:

Case 1: $\alpha_i^\ell = 0$ for some $i \in G^\ell(\mathcal{C})$. Pick $\widehat{\alpha}^\ell$ such that $\widehat{\alpha}_i^\ell = 1$ and $\widehat{\alpha}_k^\ell = \alpha_k^\ell \ \forall\ k \neq i$. First, if $W\left(\mathcal{C};\widehat{\alpha}^\ell,\alpha^{-\ell}\right) = W\left(\mathcal{C};\alpha^\ell,\alpha^{-\ell}\right)$, then $U^\ell\left(\mathcal{C};\widehat{\alpha}^\ell,\alpha^{-\ell}\right) = U^\ell\left(\mathcal{C};\alpha^\ell,\alpha^{-\ell}\right)$. Second, if $i \in W\left(\mathcal{C};\alpha^\ell,\alpha^{-\ell}\right)$ and $\left|W\left(\mathcal{C};\alpha^\ell,\alpha^{-\ell}\right)\right| \geq 2$, then $W\left(\mathcal{C};\widehat{\alpha}^\ell,\alpha^{-\ell}\right) = \{i\}$ and $U^\ell\left(\mathcal{C};\widehat{\alpha}^\ell,\alpha^{-\ell}\right) \geq U^\ell\left(\mathcal{C};\alpha^\ell,\alpha^{-\ell}\right)$ with a strict inequality when $\alpha^{-\ell}$ is such that there exists $k \in W\left(\mathcal{C};\alpha^\ell,\alpha^{-\ell}\right)$ with $k \notin G^\ell(\mathcal{C})$. Finally, if $i \in R\left(\mathcal{C};\alpha^\ell,\alpha^{-\ell}\right)$, then $W\left(\mathcal{C};\widehat{\alpha}^\ell,\alpha^{-\ell}\right) = W\left(\mathcal{C};\alpha^\ell,\alpha^{-\ell}\right) \cup \{i\}$ and $p_k\left(\mathcal{C};\widehat{\alpha}^\ell,\alpha^{-\ell}\right) < p_k\left(\mathcal{C};\alpha^\ell,\alpha^{-\ell}\right) \ \forall\ k \in W\left(\mathcal{C};\alpha^\ell,\alpha^{-\ell}\right)$. Then, $U^\ell\left(\mathcal{C};\widehat{\alpha}^\ell,\alpha^{-\ell}\right) \geq U^\ell\left(\mathcal{C};\alpha^\ell,\alpha^{-\ell}\right)$ with a strict inequality when $\alpha^{-\ell}$ is such that there exists $k \in W\left(\mathcal{C};\alpha^\ell,\alpha^{-\ell}\right)$ with $k \notin G^\ell(\mathcal{C})$.

Case 2: $\alpha_i^\ell = 1$ for some $i \in L^\ell(\mathcal{C})$. Pick $\widehat{\alpha}^\ell$ such that $\widehat{\alpha}_i^\ell = 0$ and $\widehat{\alpha}_k^\ell = \alpha_k^\ell \ \forall\ k \neq i$. First, if $W\left(\mathcal{C};\widehat{\alpha}^\ell,\alpha^{-\ell}\right) = W\left(\mathcal{C};\alpha^\ell,\alpha^{-\ell}\right)$, then $U^\ell\left(\mathcal{C};\widehat{\alpha}^\ell,\alpha^{-\ell}\right) = U^\ell\left(\mathcal{C};\alpha^\ell,\alpha^{-\ell}\right)$. Second, if $i \in W\left(\mathcal{C};\alpha^\ell,\alpha^{-\ell}\right)$ and $\left|W\left(\mathcal{C};\alpha^\ell,\alpha^{-\ell}\right)\right| \geq 2$, then $i \notin W\left(\mathcal{C};\widehat{\alpha}^\ell,\alpha^{-\ell}\right)$ and $U^\ell\left(\mathcal{C};\widehat{\alpha}^\ell,\alpha^{-\ell}\right) \geq U^\ell\left(\mathcal{C};\alpha^\ell,\alpha^{-\ell}\right)$ with a strict inequality when $\alpha^{-\ell}$ is such that there exists $k \in W\left(\mathcal{C};\widehat{\alpha}^\ell,\alpha^{-\ell}\right)$ with $k \notin L^\ell(\mathcal{C})$. Finally, if $\{i\} = W\left(\mathcal{C};\alpha^\ell,\alpha^{-\ell}\right)$ and $R\left(\mathcal{C};\alpha^\ell,\alpha^{-\ell}\right) \neq \emptyset$, then $W\left(\mathcal{C};\widehat{\alpha}^\ell,\alpha^{-\ell}\right) = W\left(\mathcal{C};\alpha^\ell,\alpha^{-\ell}\right) \cup R\left(\mathcal{C};\alpha^\ell,\alpha^{-\ell}\right)$ and $p_k\left(\mathcal{C};\widehat{\alpha}^\ell,\alpha^{-\ell}\right) > p_k\left(\mathcal{C};\alpha^\ell,\alpha^{-\ell}\right) \ \forall\ k \in W\left(\mathcal{C};\widehat{\alpha}^\ell,\alpha^{-\ell}\right), k \neq i$ and $p_i\left(\mathcal{C};\widehat{\alpha}^\ell,\alpha^{-\ell}\right) < p_i\left(\mathcal{C};\alpha^\ell,\alpha^{-\ell}\right)$. Then, $U^\ell\left(\mathcal{C};\widehat{\alpha}^\ell,\alpha^{-\ell}\right) \geq U^\ell\left(\mathcal{C};\alpha^\ell,\alpha^{-\ell}\right)$ with a strict inequality when $\alpha^{-\ell}$ is such that there exists $k \in W\left(\mathcal{C};\widehat{\alpha}^\ell,\alpha^{-\ell}\right)$ with $k \notin L^\ell(\mathcal{C})$.

(Necessity) Let $\widehat{\alpha}^\ell$ be an admissible strategy. For any arbitrary strategy $\alpha^\ell \left(\neq \widehat{\alpha}^\ell\right)$ it suffices that there exists such a strategy profile $\alpha^{-\ell}$ for $-\ell$ such that $U^\ell\left(\mathcal{C};\widehat{\alpha}^\ell,\alpha^{-\ell}\right) > U^\ell\left(\mathcal{C};\alpha^\ell,\alpha^{-\ell}\right)$. There are two cases to consider:

Case 1: $\widehat{\alpha}_i^\ell = 1$ and $\alpha_i^\ell = 0$ for some $i \in \mathcal{C}$. Since $\widehat{\alpha}^\ell$ is admissible, it must then be that $i \notin L^\ell(\mathcal{C})$. Pick any $k \in L^\ell(\mathcal{C})$. We know that $\widehat{\alpha}_k^\ell = 0$. Choose $\alpha^{-\ell}$ such that $\sum_{-\ell} \alpha_i^{-\ell} = \sum_{-\ell} \alpha_k^{-\ell}$ and $\sum_{-\ell} \alpha_j^{-\ell} < \sum_{-\ell} \alpha_i^{-\ell} - 1$ for all $j \neq i, k$. Under $\left(\alpha^\ell, \alpha^{-\ell}\right)$, there is either a tie between i and k or k wins outright. On the other hand, under $\left(\widehat{\alpha}^\ell, \alpha^{-\ell}\right)$, candidate i is the outright winner.

Case 2: $\widehat{\alpha}_i^\ell = 0$ and $\alpha_i^\ell = 1$ for some $i \in \mathcal{C}$. Since $\widehat{\alpha}^\ell$ is admissible, it must then be that $i \notin G^\ell(\mathcal{C})$. Pick any $k \in G^\ell(\mathcal{C})$. We know that $\widehat{\alpha}_k^\ell = 1$. Choose $\alpha^{-\ell}$ such that $\sum_{-\ell} \alpha_i^{-\ell} = \sum_{-\ell} \alpha_k^{-\ell}$ and $\sum_{-\ell} \alpha_j^{-\ell} < \sum_{-\ell} \alpha_i^{-\ell} - 1$ for all $j \neq i, k$. Under $\left(\alpha^\ell, \alpha^{-\ell}\right)$, there is either a tie between i and k or i wins outright. On the other hand, under $\left(\widehat{\alpha}^\ell, \alpha^{-\ell}\right)$, candidate k is the outright winner. □

Proof. (**Lemma 2**)
Suppose not. Then for each $\alpha^\ell \in BR^\ell(\mathcal{C}, \alpha^{-\ell})$ there exists an $i \in \mathcal{C}$ such that either 1) $v_i^\ell > U^\ell(\mathcal{C}; \alpha^\ell, \alpha^{-\ell})$ and $\alpha_i^\ell = 0$, or 2) $v_i^\ell < U^\ell(\mathcal{C}; \alpha^\ell, \alpha^{-\ell})$ and $\alpha_i^\ell = 1$.

Case 1: $W(\mathcal{C}; \alpha^\ell, \alpha^{-\ell}) \cup R(\mathcal{C}; \alpha^\ell, \alpha^{-\ell})$ is a singleton. In this case every strategy is a best response and so is a relatively monotonic strategy, a contradiction.

Case 2: $i \notin W(\mathcal{C}; \alpha^\ell, \alpha^{-\ell}) \cup R(\mathcal{C}; \alpha^\ell, \alpha^{-\ell})$. In this case ℓ 's payoff does not depend upon α_i^ℓ. Construct $\widetilde{\alpha}^\ell$ such that $\widetilde{\alpha}_i^\ell = 1$ for all i such that $v_i^\ell > U^\ell(\mathcal{C}; \widetilde{\alpha}^\ell, \alpha^{-\ell})$ and $\widetilde{\alpha}_i^\ell = 0$ for all i such that $v_i^\ell < U^\ell(\mathcal{C}; \widetilde{\alpha}^\ell, \alpha^{-\ell})$. Then, $\alpha^\ell \in BR^\ell(\mathcal{C}, \alpha^{-\ell})$ implies $\widetilde{\alpha}^\ell \in BR^\ell(\mathcal{C}, \alpha^{-\ell})$. Hence, the violation of relative monotonicity must take place for $i \in W(\mathcal{C}; \alpha^\ell, \alpha^{-\ell}) \cup R(\mathcal{C}; \alpha^\ell, \alpha^{-\ell})$. Let us assume that condition 1) gets violated and show a contradiction. A similar proof follows for 2).

Case 2(i): $i \in W(\mathcal{C}; \alpha^\ell, \alpha^{-\ell})$. In this case, construct a strategy such that $\widehat{\alpha}_i^\ell = 1$ and $\widehat{\alpha}_k^\ell = \alpha_k^\ell$ for $k \neq i$. ℓ gets payoff v_i^ℓ under $\widehat{\alpha}$ which is strictly greater than $U^\ell(\mathcal{C}; \alpha^\ell, \alpha^{-\ell})$, hence $\alpha^\ell \notin BR^\ell(\mathcal{C}, \alpha^{-\ell})$.

Case 2(ii): $i \in R(\mathcal{C}; \alpha^\ell, \alpha^{-\ell})$. Again, construct a strategy such that $\widehat{\alpha}_i^\ell = 1$ and $\widehat{\alpha}_k^\ell = \alpha_k^\ell$ for $k \neq i$. ℓ's payoff under $\widehat{\alpha}$ is $\frac{1}{|W|+1} v_i^\ell + \frac{|W|}{|W|+1} U^\ell(\mathcal{C}; \alpha^\ell, \alpha^{-\ell})$ which is strictly greater than $U^\ell(\mathcal{C}; \alpha^\ell, \alpha^{-\ell})$, hence $\alpha^\ell \notin BR^\ell(\mathcal{C}, \alpha^{-\ell})$. □

Proof. (**Proposition 2**)
Given a voting profile α, let $R(\mathcal{C};\alpha)$ denote the set of candidates in $\mathcal{C}$ who receive exactly one vote less than those in the set $W(\mathcal{C};\alpha)$. For $\ell \in \mathcal{N}$, suppose that $\alpha^\ell \in BR^\ell(\mathcal{C};\alpha)$ and there exists a $j \in \mathcal{C}$ such that $v_j^\ell > U^\ell(\mathcal{C};\alpha)$ and $\alpha_j^\ell = 0$.

Claim 1: $j \notin W(\mathcal{C};\alpha) \cup R(\mathcal{C};\alpha)$.

Define $\widetilde{\alpha}^\ell$ such that $\widetilde{\alpha}_j^\ell = 1$ and $\widetilde{\alpha}_k^\ell = \alpha_k^\ell$ for all $k \neq j$. To verify this claim consider the two possibilities:

(*i*) Suppose $j \in W(\mathcal{C};\alpha)$. Then $W(\mathcal{C};\widetilde{\alpha}^\ell,\alpha^{-\ell}) = \{j\}$ and hence, $v_j^\ell \equiv U^\ell(\mathcal{C};\widetilde{\alpha}^\ell,\alpha^{-\ell}) > U^\ell(\mathcal{C};\alpha)$. But then $\alpha^\ell \notin BR^\ell(\mathcal{C};\alpha)$.

(*ii*) Suppose $j \in R(\mathcal{C};\alpha)$. Then, $W(\mathcal{C};\widetilde{\alpha}^\ell,\alpha^{-\ell}) = W(\mathcal{C};\alpha) \cup \{j\}$ and hence, $U^\ell(\mathcal{C};\widetilde{\alpha}^\ell,\alpha^{-\ell}) = \frac{|W|}{|W|+1}U^\ell(\mathcal{C};\alpha)+\frac{1}{|W|+1}v^\ell j > U^\ell(\mathcal{C};\alpha)$. But then $\alpha^\ell \notin BR^\ell(\mathcal{C};\alpha)$.

Given that $j \notin W(\mathcal{C};\alpha) \cup R(\mathcal{C};\alpha)$, we have $U^\ell(\mathcal{C};\widetilde{\alpha}^\ell,\alpha^{-\ell}) = W(\mathcal{C};\alpha)$ and hence, $\widetilde{\alpha}^\ell \in BR^\ell(\mathcal{C};\alpha)$. Hence, we can replace α^ℓ with $\widetilde{\alpha}^\ell$ such that $\widetilde{\alpha}_j^\ell = 1$ for all $j \in \mathcal{C}$ satisfying $v_j^\ell > U^\ell(\mathcal{C};\alpha)$ and $\widetilde{\alpha}_j^\ell = \alpha^\ell$ otherwise and have $\widetilde{\alpha}^\ell \in BR^\ell(\mathcal{C};\alpha)$.

Now suppose that, $\alpha^\ell \in BR^\ell(\mathcal{C};\alpha)$ and there exists a $j \in \mathcal{C}$ such that $v_j^\ell < U^\ell(\mathcal{C};\alpha)$ and $\alpha_j^\ell = 1$.

Claim 2: $j \notin W(\mathcal{C};\alpha)$. The proof is analogous to that of claim 1. Hence, we can replace $\widetilde{\alpha}_j^\ell = 0$ in place of $\alpha_j^\ell = 1$ for all $j \in \mathcal{C}$ such that $v_j^\ell < U^\ell(\mathcal{C};\alpha)$ and the new strategy $\widetilde{\alpha}^\ell \in BR^\ell(\mathcal{C};\alpha)$.

We have thus constructed a relatively sincere strategy which belongs to $BR^\ell(\mathcal{C};\alpha)$.

□

Proof. (**Proposition 3**) Consider a winning set W with at least 3 elements in it. Note that this means that for each voter i, $G^i \neq L^i$. Hence, he must vote for at least one policy and NOT vote for at least one policy. For any element $t \in W$, let $F_t(D)$ denote the set of voters in the interval D who are voting for element t. Let $A \equiv \frac{1}{|W|}\sum_{t\in W} t$ denote the average policy over set W.

Claim 1: For any elements y and z such that $y < z \leq A$, $F_y([0,1]) = F_z([0,1])$ (i,e same people must be voting for both x and y.)

Proof. Since, $z \leq A$,by relative sincerity and concavity, $[0,z] \subset F_z([0,1])$ and hence, $F_y([0,z]) \subset F_z([0,z])$. Also, if $i > z$ votes for y then he must vote for z as well. Hence, $F_y((z,1]) \subset F_z((z,1])$. Hence, $F_y([0,1]) \subset F_z([0,1])$. But since $|F_y([0,1])| = |F_z([0,1])|$ (since both y and z are in W) we have $F_y([0,1]) = F_z([0,1])$ $\square$

What above claim states is, voters can be partitioned into two disjoint and exhaustive sets: those voting for ALL elements $\leq A$ and those voting for all the elements $> A$.

Claim 2 Let m be the median voter. Without any loss of generality, suppose that m votes for $h \leq A$. Then, every $i < m$ votes for h as well.

Proof. Suppose that m votes for some $h \leq A$, while i does not. Then he must be voting for element $k > A$. Then since $i < m$, we have $v_k^m > v_h^m$ (see property 2 of the preferences) but since m is voting for h by sincerity, he must vote for k as well. That means m votes for all the elements in W which gives us a contradiction. $\square$

Above claim implies that if m votes for $h \leq A$, then all the elements $\leq A$ get strictly more than half of the votes while those $> A$ get strictly less than half. This means that they can't be in W. And hence we have established that there cannot be more that two elements in W. $\square$

Proof. (**Proposition 4**) Trivial $\square$

Proof. (**Proposition 5**) (Necessity) Condition (i) follows from proposition 3 and $W(\mathcal{C}, \alpha(.)) = \mathcal{C}$.

It has to be the case that no candidate i wants to exit the race. If he chooses to deviate, i.e. $\widehat{s}_i = 0$, then $\widehat{W}(\mathcal{C}\backslash\{i\}, \alpha(.)) = \mathcal{C}\backslash\{i\}$ and he will get an expected utility $\widehat{U}_i = \frac{c_{k(i)}}{c-1} v^i_{k(i)}$, where $k(i) = 1$ if $w_i = w_2$ and 2 if $w_i = w_1$. Hence, candidate i does not want to deviate if

$$U_i \geq \widehat{U}_i \Leftrightarrow -v^i_{k(i)} \geq \frac{c\,(c-1)}{c_{k(i)}}\delta$$

It must also be the case that no other citizen with $w_i \in \{w_1, w_2\}$ wants to enter the race. Indeed, suppose $\widehat{s}_i = 1$. Then $\widehat{W}\left(\mathcal{C} \cup \{i\}, \alpha\left(.\right)\right) = \mathcal{C} \cup \{i\}$ and citizen i gets an expected utility $\widehat{U}_i = \frac{c_{k(i)}}{c+1} v^i_{k(i)} - \delta$. Rather, if he does not enter, his expected utility is $U_i = \frac{c_{k(i)}}{c} v^i_{k(i)}$. Then, citizen i does not want to enter if

$$U_i \geq \widehat{U}_i \Leftrightarrow \frac{c\,(c+1)}{c_{k(i)}}\delta \geq -v^i_{k(i)}$$

Finally, there cannot be a citizen with an ideal point different from w_1 and w_2, who is guaranteed to win outright or tie and who wants to enter the race. Remember that admissibility requires $\alpha^\ell_i = 1$ for all $\ell \in \mathcal{N}$ and $i \in \mathcal{C}$ such that $i \in G^\ell\left(\mathcal{C}\right)$. Hence, min $|N_i| = \left|\left\{\ell \in \mathcal{N} : i \in G^\ell\left(\mathcal{C}\right)\right\}\right|$ for all $i \in \mathcal{C}$. Also from Proposition 7, we have that $|N_i| = \frac{|\mathcal{N} \setminus \mathcal{M}|}{2}$ for all $i \in \mathcal{C}$ with $w_i \in \{w_1, w_2\}$. First, note that $\left|\left\{\ell \in \mathcal{N} : i \in G^\ell\left(\mathcal{C}\right)\right\}\right| < \frac{|\mathcal{N} \setminus \mathcal{M}|}{2}$ for all $i \in \mathcal{C}$ with $w_i < w_1$ since $\left\{\ell \in \mathcal{N} : i \in G^\ell\left(\mathcal{C}\right)\right\} \subset \left\{\ell \in \mathcal{N} : w_1 \geq w_\ell\right\}$ and $\left\{\ell \in \mathcal{N} : w_\ell = w_1\right\} \neq \emptyset$. In other words, any candidate with a position to the left of w_1 cannot be the most-preferred candidate of citizens at and to the right of w_1. This implies that $\left|\left\{\ell \in \mathcal{N} : i \in G^\ell\left(\mathcal{C}\right)\right\}\right| < \frac{|\mathcal{N} \setminus \mathcal{M}|}{2}$ for any such candidate. The same is true for all i such that $w_i > w_2$. However, it is not necessarily true for $i \in \mathcal{N}$ with $w_i \in (w_1, w_2)$. If such a citizen enters the race, then any candidate j at w_1 or w_2 will be the least-preferred one for the voters at the median. Hence $\alpha^m_j = 0$ and max $|N_j| = \frac{|\mathcal{N} \setminus \mathcal{M}|}{2}$ for all j with $w_j \in \{w_1, w_2\}$. Now, if $\left|\left\{\ell \in \mathcal{N} : i \in G^\ell\left(\mathcal{C}\right)\right\}\right| > \frac{|\mathcal{N} \setminus \mathcal{M}|}{2}$, then candidate i wins outright. Thus it must be him who does not want to enter, i.e.

$$\frac{1}{c}\left(c_1 v^i_1 + c_2 v^i_2\right) > v^i_i - \delta \Leftrightarrow -\frac{1}{c}\sum_{k \in \{1,2\}} c_k v^i_k < \delta$$

Rather, if $\left|\left\{\ell \in \mathcal{N} : i \in G^{\ell}\left(\mathcal{C}\right)\right\}\right| = \frac{|\mathcal{N}\backslash\mathcal{M}|}{2}$, then he will tie with the other candidates if no other citizen votes for him. It must then be that

$$\frac{1}{c}\left(c_1 v_1^i + c_2 v_2^i\right) > \frac{1}{c+1}\left(v_i^i + c_1 v_1^i + c_2 v_2^i\right) - \delta \Leftrightarrow -\frac{1}{c\left(c+1\right)} \sum_{k \in \{1,2\}} c_k v_k^i < \delta$$

(Sufficiency) It remains to show that these conditions are sufficient. Let $\alpha\left(\mathcal{C} \cup \{i\}\right)$ be a voting rule such that for all $\ell \in \mathcal{N}\backslash\mathcal{M}$,

$$\begin{aligned} \alpha_j^{\ell}\left(\mathcal{C} \cup \{i\}\right) &= \alpha_j^{\ell}\left(\mathcal{C}\right) \text{ for all } j \in \mathcal{C}, \text{ and} \\ \alpha_i^{\ell}\left(\mathcal{C} \cup \{i\}\right) &= 1 \text{ if } i \in G^{\ell}\left(\mathcal{C} \cup \{i\}\right) \text{ and 0 otherwise.} \end{aligned}$$

For all $\ell \in \mathcal{M}$, let $\alpha_j^{\ell}\left(\mathcal{C} \cup \{i\}\right) = 1$ for all $j \in \mathcal{C}$ and $\alpha_i^{\ell}\left(\mathcal{C} \cup \{i\}\right) = 0$ if $w_i \notin [w_1, w_2]$. Otherwise let $\alpha_j^{\ell}\left(\mathcal{C} \cup \{i\}\right) = 0$ for all $j \in \left(\mathcal{C} \cup \{i\}\right)$ with $w_j \in \{w_1, w_2\}$ and $\alpha_j^{\ell}\left(\mathcal{C} \cup \{i\}\right) = 1$ when $w_j \in (w_1, w_2)$.

Note that admissibility is satisfied and $\alpha\left(.\right)$ is a voting equilibrium. In addition, citizen i does not want to enter. □

Proof. (**Proposition 6**) First note that the first two necessary conditions in proposition 5 are necessary and sufficient for a two-candidate Political Equilibrium under Plurality Voting. Hence, the set of policy outcomes from Political Equilibria under Approval Voting where $W\left(\mathcal{C}, \alpha\left(.\right)\right) = \mathcal{C}$ is a subset of the set of policy outcomes from the two-candidate Political Equilibria under Plurality Voting.

Now suppose that $\{\widetilde{w}_1, \widetilde{w}_2\}$ is the policy outcome of a two-candidate Political Equilibrium under Plurality Voting. By proposition 5 (i) (which is the same under both Plurality and Approval voting), we have $\widetilde{v}_1^m = \tilde{v}_2^m$. Hence, condition ($i$) of proposition 5 holds.

The sufficient condition in proposition 5 is also satisfied. To see this, take any $k \in \mathcal{N}$ with $w_k \in (\widetilde{w}_1, \widetilde{w}_2)$. We will show that $\left\{\ell \in \mathcal{N} : k \in G^{\ell}\left(\{\widetilde{w}_1, w_k, \widetilde{w}_2\}\right)\right\} \subseteq \left\{\ell \in \mathcal{N} : k \in G^{\ell}\left(\{w_1, w_k, w_2\}\right)\right\}$. First, consider $\ell \in \mathcal{N}$ such that $k \in G^{\ell}\left(\{\widetilde{w}_1, w_k, \widetilde{w}_2\}\right)$. This implies that $v_k^{\ell} \geq \max\ \left\{\widetilde{v}_1^{\ell}, \tilde{v}_2^{\ell}\right\}$. Since $[\widetilde{w}_1, \widetilde{w}_2] \subset [w_1, w_2]$ and preferences are single-peaked and concave, we have for those citizens that $\tilde{v}_i^\ell > v_i^\ell$ for all $i \in \{1,2\}$. Hence, max $\left\{\tilde{v}_1^\ell, \tilde{v}_2^\ell\right\} >$ max $\left\{v_1^\ell, v_2^\ell\right\}$. Then, $v_k^\ell >$ max $\left\{v_1^\ell, v_2^\ell\right\}$ and $k \in G^\ell\left(\{w_1, w_k, w_2\}\right)$.

Second, consider $\ell \in \mathcal{N}$ such that $k \in G^\ell\left(\{w_1, w_k, w_2\}\right)$. If $w_\ell \notin (\widetilde{w}_1, \widetilde{w}_2)$, then either $w_k > \tilde{w}_1 \geq w_\ell$ or $w_\ell \geq \tilde{w}_2 > w_k$. In the first case, $\tilde{v}_1^\ell > v_k^\ell$ while $\tilde{v}_2^\ell > v_k^\ell$ in the second case. As a result, $k \notin G^\ell\left(\{\widetilde{w}_1, w_k, \widetilde{w}_2\}\right)$. Rather, if $w_\ell \in (\widetilde{w}_1, \widetilde{w}_2)$, then either $v_k^\ell \geq \tilde{v}_i^\ell > v_i^\ell$ for all $i \in \{1,2\}$, in which case $k \in G^\ell\left(\{\widetilde{w}_1, w_k, \widetilde{w}_2\}\right)$ or $\tilde{v}_i^\ell > v_k^\ell \geq v_i^\ell$ for some $i \in \{1,2\}$ in which case $k \notin G^\ell\left(\{\widetilde{w}_1, w_k, \widetilde{w}_2\}\right)$.

As a result, $\left\{\ell \in \mathcal{N} : k \in G^\ell\left(\{\widetilde{w}_1, w_k, \widetilde{w}_2\}\right)\right\} \subseteq \left\{\ell \in \mathcal{N} : k \in G^\ell\left(\{w_1, w_k, w_2\}\right)\right\}$ and $\mid \{\ell \in \mathcal{N} : k \in G^\ell\left(\{\widetilde{w}_1, w_k, \widetilde{w}_2\}\right)\} \mid < \frac{|\mathcal{N}\setminus\mathcal{M}|}{2} - 1$.

The same is true for condition (iv) of proposition 5 since we have shown that $\{\ell \in \mathcal{N} : k \in G^\ell\left(\{\widetilde{w}_1, w_k, \widetilde{w}_2\}\right)\} \subseteq \left\{\ell \in \mathcal{N} : k \in G^\ell\left(\{w_1, w_k, w_2\}\right)\right\}$ and we know that $\tilde{v}_i^k > v_i^k$ for all $i \in \{1,2\}$. Hence, if condition (iv) is satisfied for $\{w_1, w_2\}$, it must also hold for $\{\widetilde{w}_1, \widetilde{w}_2\}$.

It remains to show that conditions (ii) and (iii) are satisfied. We will proceed starting from the two-candidate Political Equilibrium under Plurality Voting, adding candidates at $\tilde{w}_1$ and/or $\tilde{w}_2$ until condition (iii) is satisfied, then showing that condition (ii) holds. Since $\{\widetilde{w}_1, \widetilde{w}_2\}$ is the policy outcome under Plurality Voting, we know that $\widetilde{c}_1 = \widetilde{c}_2 = 1$ and $\tilde{v} \equiv \tilde{v}_2^1 = \tilde{v}_1^2 \geq 2\delta$ (the inequality by condition (ii) of proposition 5, which is the same under Plurality voting, and the equality by symmetry). Either $\nexists\ i \in \mathcal{N}\setminus\mathcal{C}$ with $w_i \in \{\widetilde{w}_1, \widetilde{w}_2\}$ such that -$\tilde{v} > 6\delta$. Then, condition (iii) of proposition 5 is satisfied, as well as condition (ii). Suppose rather that there exists $i \in \mathcal{N}\setminus\mathcal{C}$ with $w_i \in \{\widetilde{w}_1, \widetilde{w}_2\}$ such that -$\tilde{v} > 6\delta$. In that case, this citizen chooses to enter the race and $\widetilde{c}_i = 2$, while $\widetilde{c}_{k(i)} = 1$. Now, either $12\delta \geq -\tilde{v}$ or $\widetilde{c}_i = n_i$ (where $n_i \equiv |\{\ell \in \mathcal{N} : w_\ell = w_i\}|$, i.e. all the citizens with ideal point w_i are now running) in which case no other citizen with the same ideal point wants to enter the race. Condition (iii) now holds. Also, since, we had $-\tilde{v} > 6\delta$ in the first place and that $\widetilde{c}_i = 2$ and $\widetilde{c}_{k(i)} = 1$, condition (ii) is satisfied. Thus there exists a three-candidate

Political Equilibrium under Approval Voting with $\{\widetilde{w}_1, \widetilde{w}_2\}$ as policy outcome. If rather, $-\tilde{v} > 12\delta$ and $\tilde{n}_i < c_i$ for some $i \in \{1, 2\}$, then we proceed in the same way, adding one more candidate. □

Proof. (**Lemma 3**) If there are two or less than two positions where candidates are located, then every sincere strategy is relatively sincere. Consider the non-trivial possibility if three or more positions at which candidates are situated. Consider an intermediate position t (i.e., there is at least one position to the right of t and one to the left of t). Consider the following voting strategy- for each citizen j, $\alpha_i^j = 1$ if and only if $v_i^j \geq v_t^j$. Such a strategy is admissible for each i. It is easy to see that under this strategy profile the candidate/s at position t get/s at least two votes more than any other alternatives. Hence it is a voting equilibrium. The voting strategies are, by construction, relatively sincere. □

Proof. (**Proposition 7**) Consider 1-candidate Political Equilibria. Note first that there exists an equilibrium in which the median citizen runs unopposed. Indeed, by assumption, $-v_{x_0}^m \geq \delta$ which corresponds to condition (i) of proposition 4. Moreover, no other citizen h with $w_h = m$ wants to enter the race, while for all $h \in \mathcal{N} \backslash \mathcal{C}$ with $w_h \neq m$, we have $v_h^m < v_m^m = 0$ (by single-peakedness), which satisfy condition (ii) of the same proposition.

Second, note that there cannot exist a Political Equilibrium in which citizen i with $-v_i^m \geq \delta$ runs unopposed. Indeed, if the median citizen enters, he will win outright and get a utility $\widehat{U}_m = v_m^m - \delta = -\delta$. If rather he does not enter, he gets a utility $U_m = v_i^m$. Then he will want to enter since $\widehat{U}_m \geq U_m$, a contradiction. Hence, in any 1-candidate Political Equilibrium (and we have shown that there exists at least one), candidate i 's position must be such that $-v_i^m < \delta$.

Now consider Political Equilibria with at least two candidates. By proposition 3, we know that candidates are located at exactly two positions, w_1 and w_2, with $v_1^m = v_2^m$. Let without loss of generality $m \geq \overline{w}$, where $\overline{w} = \frac{1}{c}(c_1 w_1 + c_2 w_2)$ (a similar argument would apply for $\overline{w} \geq m$). Now, for all $\ell \in \mathcal{N}$,

$$v_{\overline{w}}^{\ell} \geq \frac{1}{c}\left(c_1 v_1^{\ell} + c_2 v_2^{\ell}\right)$$

by Jensen's inequality. Since $m \geq \overline{w}$, we have $v_m^{\ell} \geq v_{\overline{w}}^{\ell}$ for all $\ell \in \mathcal{N}$ with $w_{\ell} \geq m$. Hence,

$$v_m^{\ell} \geq \frac{1}{c}\left(c_1 v_1^{\ell} + c_2 v_2^{\ell}\right)$$

for all such citizens. Then, if a citizen with the median ideal point decides to enter, he will get a vote total $|N_m| \geq \frac{|\mathcal{N} \setminus \mathcal{M}|}{2} + |\mathcal{M}|$, while $|N_1|$ and $|N_2|$ will be at most $\frac{|\mathcal{N} \setminus \mathcal{M}|}{2}$ (since $\{1\} \in L^{\ell}(\{1, 2, m\})$ for all $\ell \in \mathcal{N}$ with $w_{\ell} \geq m$ and $\{2\} \in L^{\ell}(\{1, 2, m\})$ for all $\ell \in \mathcal{N}$ with $m \geq w_{\ell}$.) Hence, if a citizen at the median enters the race, he is going to win outright. Now, he will want to enter if

$$v_m^m - \delta \geq \frac{1}{c}\left(c_1 v_1^m + c_2 v_2^m\right) \Leftrightarrow -v_i^m \geq \delta$$

where $i \in \{1, 2\}$ (the second inequality comes from $v_m^m = 0$ and $v_1^m = v_2^m$).

As a result, it must be that $-v_i^m < \delta$ for all $i \in \mathcal{C}$ for m not to enter the race. $\square$

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